

Uniform and Participation-expanding Reforms in Decentralized Redistribution: Who Gains and Who Loses?*

Alireza Mohammadi Sepahvand[†]

November 13, 2025

Abstract

Across many settings, from social transfers to education and health services, reforms that aim to make access uniformly easier for everyone often exacerbate inequality. When access to the redistributive surplus is determined through competition, such reforms increase players' efficiency in the contest but also raise the endogenous efficiency threshold for participation. I formalize this threshold-shifting mechanism in a model of decentralized redistribution, where heterogeneous efficiencies determine both who competes and how the redistributive surplus is allocated. The model delivers two main results. First, a uniform efficiency gain—i.e., equal reductions in access costs—does not expand participation. Instead, it amplifies inequality among participants by reinforcing the relative efficiency of those who were already highly efficient prior to the reform. Yet, such a reform also reduces contest-induced welfare loss, thereby giving rise to an equity–efficiency trade-off for policymakers. Second, a participation-expanding reform that brings excluded players into the contest can reduce both inequality and welfare loss, particularly when: (i) the new entrants are sufficiently efficient relative to the average participant; and (ii) the Herfindahl–Hirschman Index (HHI) of endogenous contest strengths—measuring the extent to which players' efficiencies exceed the participation threshold—is sufficiently high before the reform; for example, when a few strong participants dominate the competition for redistributive surplus.

Keywords: Redistribution, Inequality, Reforms, Inefficiency

JEL Codes: D72, D74, D31, C72

*All errors and omissions remain my responsibility.

[†]Department of Economics, University of Edinburgh, UK. Email: amohamma@ed.ac.uk

1 Introduction

To promote fairness in access to public resources, governments frequently introduce *uniform reforms*, i.e., reforms that reduce access costs equally for everyone—from digitized applications and simplified procedures to transparency portals. Yet these interventions have increased inequality across many contexts.

For example, in India, the introduction of biometric smartcards under Aadhaar improved payment efficiency and reduced leakage but also generated exclusion errors, especially among rural and elderly beneficiaries (Muralidharan et al. (2016)). Broader financial management reforms likewise enhanced accountability yet produced uneven benefits across regions and social groups (Banerjee et al. (2020)). In Uganda, transparency campaigns designed to curb local capture of school grants successfully reduced corruption, but the gains were concentrated in areas with better media access (Reinikka and Svensson (2004)). Why do policies meant to equalize access sometimes reinforce disparities?

Some theoretical studies show that treating heterogeneous agents symmetrically need not equalize outcomes. For instance, Moldovanu and Sela (2001) show that in contest environments with heterogeneous costs, even symmetric prize structures can induce unequal effort and inefficient dissipation; in collective-action settings, group structure and prize characteristics shape success asymmetrically (Esteban and Ray (1999); Esteban and Ray (2001)).

I isolate a simple yet important channel—the *threshold-shifting mechanism*—that helps explain the inequality-amplifying effects of uniform reforms, a novel attempt in the contest literature to the best of my knowledge. I formalize this mechanism in a tractable model of contest-based redistribution with endogenous participation, where players’ efficiencies in converting resources into effort determine the allocation of the redistributive surplus. I then derive comparative statics to study how efficiency-enhancing reforms affect the endogenous entry rule for participation (i.e., exerting positive effort) and how this change reshapes the final allocation of the surplus.

The model delivers two core results. First, a uniform efficiency improvement amplifies inequality, as it fails to expand participation and disproportionately increases the relative advantage of the most efficient players, even as all agents become more efficient under the uniform reform. By contrast, a *participation-expanding policy* that brings marginal players into the contest can simultaneously lower inequality and reduce dissipative effort, especially when pre-entry efficiency is highly concentrated or when the entrant is sufficiently efficient compared to the average of existing participants.

These results arise from simple yet fairly general economic forces. As in any model with endogenous participation, only those players in the model whose inefficiency is lower than a scaled average of the other participants’ inefficiencies endogenously choose to compete for redistribution—that is, they exert strictly positive effort. Participation is therefore

comparative and self-referential: each agent enters only if sufficiently efficient relative to others, making the participant set the fixed point of this entry rule. This threshold motivates the equilibrium notion of *contest strength*: for a participant, it is defined as the gap between the participation cutoff inefficiency and the participant’s own efficiency, and it is zero for nonparticipants. In equilibrium, redistribution shares vary with the square of relative contest strength. While this quadratic mapping amplifies differences in relative strength, the qualitative mechanisms I study do not rely on convexity.

When a uniform reform reduces everyone’s inefficiency by the same amount, two changes occur. First, the participation cutoff tightens because the collective improvement of many rivals outweighs any one individual’s gain. This, in turn, makes participation more selective and can push marginal participants out of the contest.¹ Beyond the potential exclusion of weaker participants, uniform efficiency-enhancing reforms also reshape the relative contest strengths of those who remain active. Since the bar for participation rises faster than any individual’s efficiency, every active player experiences the same absolute reduction in contest strength—that is, their efficiency advantage over the cutoff narrows by an equal amount. Because redistribution outcomes depend on relative contest strength, this uniform reduction penalizes marginal participants proportionally more than dominant ones, thereby amplifying the advantages of high-efficiency participants.

Furthermore, a uniform policy reduces aggregate dissipation from effort. Aggregate dissipation is derived endogenously in the model as a fixed share of the redistributive surplus, equal to $1 - \text{HHI}$, where the Herfindahl–Hirschman Index (HHI) measures the concentration of contest strength among active players.² The term $1 - \text{HHI}$ therefore represents the fraction of the surplus lost through strategic effort. When contest strength is evenly distributed, HHI is low and dissipation is high; and when contest strength is concentrated among a few dominant players, HHI is high and dissipation falls.³ A uniform efficiency reform tightens competition and enhances the relative advantage of the most efficient participants, so that contest strength becomes more concentrated among fewer players and aggregate dissipation declines. Hence, a uniform policy inherently entails a trade-off between equality and efficiency.

Alternatively, a participation-expanding reform that enables previously non-participants to enter the competition reduces inequality between participants and non-participants. More importantly, it can also lower aggregate dissipation—when the new entrants become sufficiently efficient relative to the average efficiency of existing participants and/or when contest strength is already highly concentrated among a few players. This is because when a new participant enters, the contest reshapes itself. To enter, a new participant

¹In this environment, there is no equilibrium with zero or only one active participant.

²Formally, $\text{HHI} = \sum_{i \in \mathcal{L}} s_i^2$, where s_i is player i ’s contest share in equilibrium; $1 - \text{HHI}$ thus captures the fraction of the surplus dissipated through effort.

³In the symmetric benchmark with L active players (all equally efficient), the concentration index is $\text{HHI} = 1/L$, so the equilibrium dissipation factor is $1 - \text{HHI} = 1 - \frac{1}{L}$.

must become more efficient than the current participation cutoff. This, in turn, reduces the inefficiency threshold which is a scaled average of the efficiencies of those already active. Thus, entry intensifies competition: existing participants lose relative strength, their shares of the surplus shrink, and the entrant gains a positive share.

The effect of entry on aggregate dissipation is more subtle and depends on the entrant's contest strength and pre-entry dissipation of the competition. Entry directly increases rivalry, which tends to raise dissipation, yet it also changes the balance of contest strength among participants. Since the new, lower inefficiency cutoff reduces every participant's contest strength by the same amount, weaker participants lose relatively more than stronger ones. This discourages weaker participants from exerting effort, sometimes enough to offset the extra rivalry created by entry. In particular, this occurs when the entrant is highly efficient or when the contest is already dominated by a few strong players. In such cases, aggregate dissipation could decline after entry, as marginal participants pull back more sharply from costly competition. This contrasts with the naive intuition that adding a stronger competitor to the contest-based redistribution necessarily increases aggregate effort dissipation.

Although the main analysis of the paper focuses on inequality in redistribution shares, it also speaks to how reforms affect wealth inequality when the alignment between wealth and efficiency becomes central. This alignment is less important in redistribution when pre-existing wealth gaps are small, when redistribution draws on large exogenous resources such as public funds or natural rents, or when investment in efficiency is slow—as in short-run redistribution. It becomes crucial, however, when redistributive resources arise endogenously from agents' wealth or when players can invest to enhance efficiency.

When wealth and efficiency are positively aligned—a *diverging-inequality regime*—the threshold-shifting mechanism remains but becomes self-reinforcing. Efficient agents are also wealthy, allowing them to sustain higher effort and stay well above the participation threshold. Uniform reforms that lower inefficiency or access costs therefore benefit these already advantaged players the most: they tighten the participation cutoff, exclude marginal entrants, and concentrate contest strength among efficient-wealthy agents. Inequality rises while aggregate dissipation falls—an equity-efficiency trade-off characteristic of a diverging regime.

When wealth and efficiency are negatively aligned—a *self-correcting regime*—the mechanism reverses. Efficient agents are relatively poor, while wealthier players are less effective competitors. Uniform reforms now strengthen the relative position of efficient low-wealth agents, allowing them to capture more of the surplus and compress inequality. By contrast, participation-expanding reforms that relax entry for wealthy but inefficient players shift surplus upward and widen disparities.

Hence, identical efficiency-enhancing reforms can yield opposite distributive outcomes depending on the sign of wealth-efficiency alignment. In positively aligned environments,

uniform reforms must be paired with inclusion-oriented measures to prevent elite capture. Where efficiency is concentrated among the less wealthy, uniform reforms can simultaneously reduce inequality and dissipation, while participation-expanding policies no longer promote equality.

1.1 Related Literature

This paper contributes to a growing literature at the intersection of public economics, development, and political economy. Its central insight—that uniform reforms can unintentionally amplify inequality through the endogenous participation—builds on and extends three strands of research.

First, it relates to the literature on contests and rent seeking, where agents exert costly effort to capture a surplus (e.g., Tullock et al. (1980); Skaperdas (1996); Konrad (2009)). Classical models in this tradition typically assume fixed participation and symmetric behavior, while subsequent work introduces heterogeneity in costs or abilities to study how these differences shape entry, rent dissipation, and equilibrium effort allocation (e.g., Moldovanu and Sela (2006); Esteban and Ray (2011)). While this literature generally treats participation thresholds as technologically or exogenously determined, a strand of models examines endogenous participation or entry cutoffs, showing that players with sufficiently low costs or high abilities self-select into competition, while others optimally abstain (e.g., Gradstein (1995); Nti (1999)). This paper builds on this literature by studying how policy-driven changes in the participation threshold affect inequality and aggregate dissipation of effort. To the best of my knowledge, this is the first model that studies how policy-driven threshold-shifting mechanisms influence equilibrium allocations in contests.

Second, the paper connects to empirical and theoretical work on access to public resources in unequal institutional environments. Across diverse settings, uniform administrative or technological upgrades—intended to lower average frictions—have disproportionately benefited already advantaged actors. In Uganda, transparency campaigns improved school funding oversight mainly for politically connected principals who could act on the disclosed information (Reinikka and Svensson (2004); Bardhan and Mookherjee (2006)). In India, welfare digitization accelerated payments for urban and digitally literate households while excluding rural claimants with limited connectivity (Dutta et al. (2014)). Similar mechanisms arise in other contexts: anti-corruption drives or e-governance reforms often improve efficiency on average but amplify gaps in access when implementation requires information, literacy, or bureaucratic familiarity (Olken (2007); Acemoglu and Robinson (2008); Olken and Pande (2012)).

Recent evidence extends this pattern beyond traditional redistributive programs. For instance, Fonseca and Matray (2024) show that financial inclusion initiatives in Brazil

increased average access to credit but widened income inequality, as more educated individuals captured disproportionate benefits. Tu et al. (2025) document that unequal access to infrastructure and services is strongly associated with health and socioeconomic inequality worldwide, underscoring how ostensibly universal improvements can translate into regressive outcomes when baseline asymmetries persist.

The second result shows that participation-expanding reforms—those that bring marginal actors into the contest—can simultaneously reduce inequality and dissipation, particularly when contest capacity is highly concentrated among a few participants or when the reform enables new entrants to compete effectively. Evidence from inclusion-oriented interventions supports this prediction: programs that facilitate entry for previously excluded groups—such as localized monitoring, targeted awareness campaigns, or participation subsidies—have been shown to expand effective participation and compress inequality (e.g., Bardhan and Mookherjee (2000); Björkman and Svensson (2009); Casey et al. (2012); Banerjee et al. (2015); Banerjee et al. (2019); Andrews et al. (2021)). These empirical patterns highlight a broader policy implication: efficiency gains that are not coupled with inclusion risk reallocating rents upward, even when formal access rules appear equal.

Finally, this paper relates to studies that examine how the correlation between wealth and efficiency affects inequality. When wealth and productive efficiency are positively aligned, improvements in efficiency can increase inequality by concentrating resources among already advantaged agents. This mechanism is similar to Aghion and Bolton (1997), where credit constraints link investment to existing wealth, and to Galor and Moav (2004) and Aghion et al. (2015), where economies move from inequality-amplifying to inequality-reducing stages as the wealth–efficiency correlation changes. Some recent studies also show that greater market power or heterogeneity in returns can strengthen this alignment and widen inequality (Aghion et al. (2023); Daminato and Pistaferri (2024); Impullitti and Rendahl (2025)). A related pattern appears in Esteban and Ray (1999), where stronger wealth–ability alignment increases conflict and inequality. Empirical evidence shows a similar pattern, with stronger capital–labor correlations and diverging wealth–income distributions associated with persistent inequality in advanced economies (Piketty and Saez (2003); Gabaix et al. (2016); Gaillard et al. (2023)). This paper sheds light on how identical efficiency-enhancing reforms can have opposite effects on inequality, depending on the alignment between wealth and efficiency.

1.2 Evidence from MGNREGA

The Mahatma Gandhi National Rural Employment Guarantee Act (MGNREGA) is a nationwide rural public-works program in India that guarantees up to 100 days of wage employment per rural household each financial year. The scheme rolled out in phases beginning in 2006 and reached all rural districts by 2008. Using district–year

administrative data, I examine how the composition of participation evolves across social groups—Scheduled Castes (SC), Scheduled Tribes (ST), and Non-SC/ST—focusing on the “households worked” share, defined for each group g as

$$y_{it}^{(g)} = \frac{\text{HH worked}_{it}^{(g)}}{\text{HH worked}_{it}^{\text{total}}} \in [0, 1], \quad (1.1)$$

for district i in year t .

The Scheduled Castes (SC) comprise communities historically subject to caste-based exclusion and occupational segregation under the Hindu caste system, many of whom were formerly regarded as “untouchables.” The Scheduled Tribes (ST) consist of indigenous and often geographically isolated groups with distinct cultural practices, who have historically faced economic marginalization and limited access to state institutions. The residual Non-SC/ST category encompasses the remaining rural population, including dominant agrarian castes and other social groups that—relative to SC and ST—have typically had greater access to land, education, and political representation.

For each group g , I partial out time-invariant district heterogeneity and observed covariates by estimating

$$y_{it}^{(g)} = \alpha_i + \mathbf{X}_{it}'\beta + \varepsilon_{it}^{(g)}, \quad (1.2)$$

with standard errors clustered at the district level, where α_i are district fixed effects and $\mathbf{X}_{it}$ is the full covariate set. I then recover the common time path by averaging the residuals by year,

$$\hat{\varepsilon}_t^{(g)} = \frac{1}{N_t} \sum_i \varepsilon_{it}^{(g)}, \quad (1.3)$$

and center the resulting series so its mean over t is zero. This procedure is equivalent to estimating year effects up to an additive constant, while not imposing year dummies so that common time variation is preserved in the plotted series.

Figure 1 plots adjusted time paths of the households-worked shares for Scheduled Castes (SC), Scheduled Tribes (ST), and Non-SC/ST groups. The shaded band marks the program’s digitization phase (e-FMS, approximately 2012–2015), and vertical dash-dot lines indicate subsequent payment milestones (DBT in 2013, Aadhaar/APBS plan in late 2013, and NeFMS in 2018). The principal pattern is a sustained rise in the Non-SC/ST share beginning around 2015–2016 that persists through and beyond the NeFMS period, while the SC share declines and the ST share remains comparatively flat.

The adjusted person-days shares show a similar pattern. During the digitization phase (2012–2015), which overlaps with DBT (2013) and Aadhaar/APBS (late 2013), composition changes across groups remain modest and unsynchronized. From 2015–2016 onward, the Non-SC/ST share rises persistently through the NeFMS milestone (2018) and beyond, while the SC share declines and the ST share remains relatively flat, with at

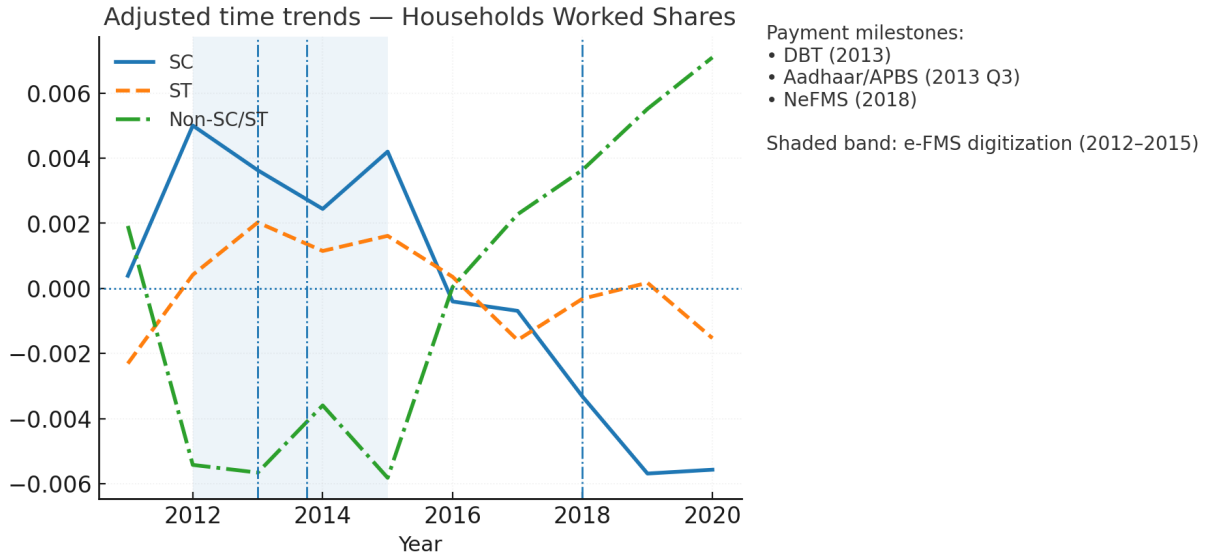


Figure 1: Adjusted time trends of households-worked shares by group (SC, ST, Non-SC/ST). Each series is district-demeaned and covariate-adjusted, averaged by calendar year and centered; vertical dash-dot lines mark payment milestones (DBT 2013; Aadhaar/APBS 2013 Q3; NeFMS 2018), and the shaded band denotes e-FMS digitization (2012–2015).

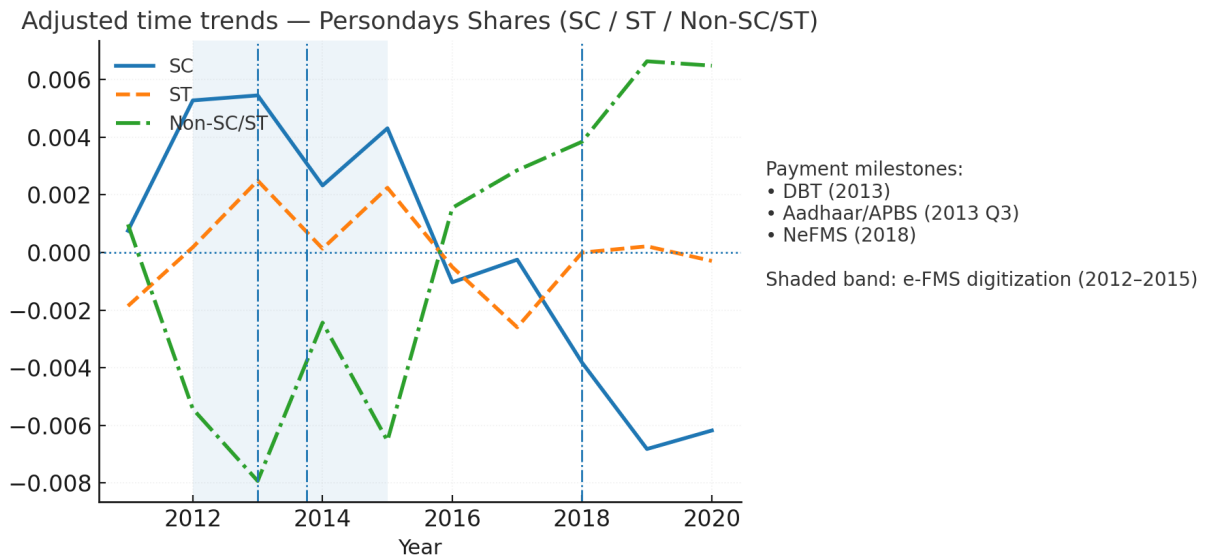


Figure 2: Adjusted time trends of person-days shares by group (SC, ST, Non-SC/ST). Each series is district-demeaned and covariate-adjusted, averaged by year and centered; vertical dash-dot lines mark DBT (2013), Aadhaar/APBS (2013 Q3), and NeFMS (2018); the shaded band denotes e-FMS digitization (2012–2015).

most a mild recovery.

The divergence that emerges after digitization—marked by rising participation of relatively advantaged groups and stagnation or decline among historically disadvantaged ones—reflects a broader pattern discussed in the section 1.1 on uniform reforms and access inequality.

2 Model

In this section, I develop a simple and tractable contest model with a lottery contest function that forms the basis for the policy analysis. I later show in 2.3 that the counterintuitive results of the policy analysis are qualitatively robust across a broad class of environments.

A finite set of players $\mathcal{N} = \{1, \dots, N\}$, each endowed with initial wealth $w_i > 0$, simultaneously exerts non-negative effort $e_i \geq 0$ to influence the distribution of a common surplus, which is determined both endogenously from the players' wealth and exogenously through a prize. The players are heterogeneous in terms of the costs they incur to adopt efforts. players face a tradeoff between spending costly effort to secure a larger share of the redistributable surplus and conserving their wealth by avoiding effort and contest participation. I first formalize how cost heterogeneity shapes strategic effort, drives endogenous exclusion, and influences post-contest inequality and aggregate dissipation. Building on this, the central analysis of the paper examines how a uniform reduction in effort costs alters participation, inequality, and aggregate dissipation.

In the benchmark analysis, I employ a lottery contest success function for tractability, and in Section 2.3 I show that the results extend to a broad class of utility functions. Specifically, player i 's share of the contested pool is given by

$$\frac{e_i}{E_N},$$

where $E_N := \sum_j e_j$ denotes total effort. This contest success function, first introduced by Tullock et al. (1980) and later microfounded by ? and ?, characterizes environments with high uncertainty in which influence scales linearly with input. Put differently, each unit of effort is treated as a lottery ticket in the redistribution process. Finally, while this formulation does not yield an equilibrium when no player exerts positive effort, in such a case, I impose the convention that

$$\frac{e_i}{E_N} := \frac{1}{N} \quad \text{and} \quad \frac{\sum_{j \neq i} e_j}{E_N} := \frac{N-1}{N}.$$

To isolate the role of cost heterogeneity, I adopt a linear cost structure that maintains tractability while preserving the key strategic trade-offs. The cost of effort in this model has two components: a fraction $\rho \in [0, 1]$ of the effort is deducted directly from the wealth, capturing institutional frictions such as bribes, delays, or processing fees. After exerting effort, the residual wealth is given by $R_i := w_i - \rho e_i$. The other component is the private cost incurred by player i , $\kappa_i e_i$. This may reflect constraints such as time, opportunity cost, or risk. The total marginal cost of effort, $\kappa_i + \rho$, shapes each player's incentive to engage in the contest. Although not required for the formal analysis, distinguishing between private and common cost parameters helps interpret the results more clearly from a policy

perspective.

As in standard contest models, I study environments where redistribution is decentralized and driven by strategic effort. Under a uniform rule, the contestable pool has two parts: (i) an *endogenous* component equal to a fixed fraction $\phi \in (0, 1]$ of aggregate residual wealth $S_R := \sum_j R_j$, and (ii) an *exogenous* prize V . Each player's expected utility is

$$U_i(e_i, e_{-i}) = \underbrace{\frac{e_i}{E_N}}_{\text{win prob.}} \left[\underbrace{R_i}_{\text{own residual}} + \underbrace{\phi \sum_{j \neq i} R_j + V}_{\text{contested pool}} \right] + \underbrace{\frac{\sum_{j \neq i} e_j}{E_N}}_{\text{lose prob.}} \underbrace{(1 - \phi) R_i}_{\text{kept if lose}} - \underbrace{\kappa_i e_i}_{\text{private cost}} \quad (2.1)$$

In expression 2.1, a player's effort influences payoffs in two ways. First, it increases the probability of gaining from others' contestable surplus, $\phi \sum_{j \neq i} R_j + V$, while also securing her own residual wealth, R_i . Second, by lowering $\frac{\sum_{j \neq i} e_j}{E_N}$, it reduces the probability of losing the contest, which would otherwise entail giving up a ϕ -share of her own residual wealth. Hence, expression 2.1 can be interpreted as the expected post-contest wealth of player i . Rearranging terms, the utility function can be expressed as

$$U_i(e_i, e_{-i}) = \underbrace{(1 - \phi) R_i}_{\text{Secured wealth}} + \underbrace{\frac{e_i}{E_N} (\phi S_R + V)}_{\text{Contested gains}} - \underbrace{\kappa_i e_i}_{\text{Private cost of effort}} \quad (2.2)$$

where $S_R := \sum_{i \in N} R_i$. This expression aligns with standard formulations in contest theory, where players trade off a contestable surplus against private costs. Under the uniform redistribution rule, a fixed fraction of aggregate residual wealth is designated as contestable, yielding the pool $\phi S_R + V$. This surplus is allocated among players in proportion to their relative efforts.⁴

Effort therefore serves a dual role: it increases a player's share of the contestable surplus and protects her own contestable residual wealth, ϕR_i , which is included in $\phi S_R + V$.

I abstract from individual liquidity constraints by assuming that the cost side does not depend on initial wealth (i.e., $h(\cdot)$ and a_i are independent of w_i), so players' effort choices are not directly constrained by their own wealth.⁵ This lets us focus on how policy-driven changes in micro-level contest efficiency shape redistribution and aggregate dissipation.

⁴Substituting $R_i = w_i - \rho e_i$ gives $S_R = S_W - \rho \sum_i e_i$, so the contestable pool $\phi S_R + V = \phi(S_W - \rho \sum_i e_i) + V$. Because the $-\rho e_i$ terms cancel in equilibrium, the effective prize is constant and equals $P = \phi S_W + V$, confirming strategic equivalence with a standard Tullock contest featuring a fixed prize.

⁵This assumption excludes wealth from the *cost* side, not from the *prize* side. Effort can still scale with aggregate wealth through $P = \phi S_W + V$ (cf. Proposition 1(ii)), and hence indirectly with each player's own wealth via S_W .

2.1 Equilibrium Characterization

This section begins by characterizing the static equilibrium of the game and the endogenous participation. Let us first introduce players' inefficiency in the contest.

Definition 1 (Inefficiency). *The inefficiency of player i is defined as*

$$a_i := \kappa_i + \rho,$$

where κ_i denotes the player's marginal cost and ρ is the institutional friction.

The cost parameter $a_i = \kappa_i + \rho$ reflects both private friction and institutional aggregate dissipation. Lower values of a_i indicate greater efficiency in converting effort into influence. I will show that in equilibrium, only players with sufficiently low a_i choose to participate. The endogenous participation margin $\tilde{a}$ and the active set $\mathcal{L}$ —the set of players who exert positive effort—are formally characterized as follows.

Definition 2 (Active Set). *The active set $\mathcal{L} \subseteq \mathcal{N}$ consists of players who exert positive effort in equilibrium, as characterized in Proposition 1. A player $i \in \mathcal{N}$ is active if and only if its contest efficiency a_i lies below the endogenous threshold:*

$$\mathcal{L} := \{i \in \mathcal{N} : a_i < \tilde{a}\}, \quad \text{where} \quad \tilde{a} := \frac{1}{L-1} \sum_{j \in \mathcal{L}} a_j.$$

I denote by $L := |\mathcal{L}|$ the number of active players.

While most contest models allow for cost heterogeneity but assume full participation, a strand of the literature endogenizes participation by showing that players with sufficiently high costs may optimally abstain, giving rise to a threshold below which only efficient players compete. The contribution of this paper is to highlight that this threshold is policy-dependent, and its endogenous shifts underpin novel comparative statics for reform design—a dimension, to the best of my knowledge, absent in prior work.

The inefficiency threshold $\tilde{a}$ in Definition 2 is self-referential. As shown in the following lemma, this is a simple fixed-point problem that admits a unique threshold and active set for any initial distribution of inefficiencies, which is a subset of players with the lowest inefficiencies.

Lemma 1 (Existence and Uniqueness of the Active Set and Threshold). *Let $\mathcal{N}$ be a finite population of $N \geq 2$ players with inefficiencies $\{a_i\}_{i \in \mathcal{N}}$. Then, there exists a unique threshold $\tilde{a} \in \mathbb{R}$ and a unique non-empty active set $\mathcal{L} \subseteq \mathcal{N}$ with $L \geq 2$ such that:*

$$\tilde{a} = \frac{1}{L-1} \sum_{j \in \mathcal{L}} a_j \quad \text{and} \quad \mathcal{L} = \{i \in \mathcal{N} : a_i < \tilde{a}\}$$

Before characterizing the equilibrium formally, I first define the contest strength of individuals given the initial inefficiencies $\{a_i\}_{i \in \mathcal{N}}$ and active set $\mathcal{L}$.

Definition 3 (Contest Strength). *For each player $i \in \mathcal{N}$, the contest strength is*

$$c_i := \max\{\tilde{a} - a_i, 0\},$$

where $\tilde{a}$ is the participation cutoff. The total contest strength is

$$\mathcal{C} := \sum_{j \in \mathcal{L}} c_j,$$

with $\mathcal{L} \subseteq \mathcal{N}$ the active set.⁶

Finally, I proceed with the first result, which establishes that, for any distribution of wealth and inefficiencies, the game admits a unique pure-strategy Nash equilibrium in which participation is determined by the active set $\mathcal{L}$ defined above.

Proposition 1 (Equilibrium Participation, Effort, and Utility). *There exists a unique pure-strategy Nash equilibrium, characterized as follows:*

(i) (Participation) *player $i \in \mathcal{N}$ exerts positive effort if and only if $i \in \mathcal{L}$, where the active set $\mathcal{L}$ and threshold $\tilde{a}$ are uniquely defined as in Definition 2.*

(ii) (Effort) *The equilibrium effort for each player i is given by:*

$$e_i = \begin{cases} \left(\frac{\tilde{a} - a_i}{\tilde{a}^2}\right) (\phi S_W + V) & \text{if } i \in \mathcal{L} \\ 0 & \text{if } i \notin \mathcal{L} \end{cases} \quad (2.3)$$

(iii) (Utility) *Each player's equilibrium utility is:*

$$U_i = (1 - \phi)w_i + s_i(\phi S_W + V) \quad (2.4)$$

where the net redistribution share is $s_i := \left(\frac{c_i}{\mathcal{C}}\right)^2$, with c_i and $\mathcal{C}$ as in Definition 3.⁷

The proof of Proposition 1 is provided in Appendix 4.

⁶In equilibrium, these objects coincide:

$$\mathcal{C} = \sum_{j \in \mathcal{L}} (\tilde{a} - a_j) = \tilde{a},$$

as shown in the proof of Proposition 1.

⁷Here $s_i := \left(c_i/\mathcal{C}\right)^2$ denotes the net fraction of the contestable surplus that accrues to player i in equilibrium (after endogenous effort costs), whereas $c_i/\mathcal{C}$ is the gross probability share of the prize before costs.

Part (i), together with Definition 1 and Lemma 1 first implies that no equilibrium exists with exactly one active player. This follows by a contradiction. Suppose there is only one player i exerting a strictly positive effort, $e_i = \varepsilon > 0$, while all other players exert zero efforts. Then player i can strictly benefit from decreasing their effort to some $0 < \varepsilon' < \varepsilon$, since it yields the same payoff at a lower cost. Moreover, the case where all players exert zero effort cannot be an equilibrium, since any player would have a profitable deviation by exerting a small effort $\varepsilon > 0$ and capturing the entire contestable surplus, which exceeds their $\frac{1}{N}$ share of the baseline payoff without exerting effort. Moreover, Part (i) states that a player i enters the contest if and only if

$$a_i < \frac{1}{L-1} \sum_{j \in \mathcal{L}} a_j$$

or, when $L > 2$, this is equivalent to

$$a_i < \frac{1}{L-2} \sum_{j \in \mathcal{L} \setminus \{i\}} a_j$$

This means that in equilibrium, each player enters the contest if and only if her inefficiency is lower than a ‘scaled average inefficiency’ of the other participants. Thus, access to redistribution is determined solely by a player’s relative efficiency, and exclusion arises endogenously through strategic self-selection.

Furthermore, Part (ii) states that in equilibrium, the effort of active players increases linearly with each player’s contest strength within the active set, that is, $\tilde{a} - a_i$, while inactive players exert no effort. Finally, Part (iii) states that in equilibrium, any player i incurs a baseline cost of exposure to redistribution, ϕw_i .

Remark 1. [Difference in redistribution shares Among Actives] The difference in redistribution shares between any two active players $i, j \in \mathcal{L}$ is given by

$$\Delta s^{ij} := s_i - s_j = \frac{2(a_j - a_i) \left(\tilde{a} - \frac{a_i + a_j}{2} \right)}{\mathcal{C}^2} \quad (2.5)$$

where the term $\left(\tilde{a} - \frac{a_i + a_j}{2} \right)$ serves as a convexity multiplier. When $a_i, a_j \ll \tilde{a}$ —that is, when both players are highly efficient, or when the pair is much more efficient than the threshold $\tilde{a}$ —even small differences in inefficiency translate into disproportionately large differences in redistribution shares. Moreover, $s_j < s_i$ if and only if $a_j > a_i$; hence, the equilibrium ranking of redistribution shares mirrors the ranking of efficiencies. Finally, in weaker contests—characterized by lower total contest strength $\mathcal{C}$ —the difference in redistribution shares between highly efficient players is further amplified.

Another aspect of contest-based redistribution is endogenous aggregate dissipation arising from strategic effort. This captures the portion of the aggregate surplus lost as

players expend resources to influence allocation outcomes. Formally, aggregate dissipation is defined as the gap between the total available surplus, including initial wealth and any exogenous transfers, and the aggregate post-contest wealth.⁸ The following corollary characterizes the equilibrium aggregate dissipation.

Corollary 1 (Equilibrium aggregate dissipation). *In equilibrium, the aggregate dissipation is given by*

$$\mathcal{D} = \delta(\phi S_W + V), \quad \text{where} \quad \delta := 1 - \frac{\sum_{i \in \mathcal{L}} c_i^2}{(\sum_{i \in \mathcal{L}} c_i)^2} = 1 - \text{HHI}(c) \in [0, 1] \quad (2.6)$$

Importantly, strategic aggregate dissipation depends on the distribution of contest strength among active players. The Herfindahl–Hirschman Index (HHI) arises endogenously in the expression for the dissipation factor δ , which measures the concentration of contest strength within the active set. Greater concentration allows dominant players to sustain their advantage with less effort, reducing aggregate dissipation. By contrast, when contest strength is more evenly distributed, rivalry intensifies, leading to higher aggregate effort and greater aggregate dissipation.

The final takeaway is that even under the linear-linear payoffs, who loses and who wins the redistribution hinges on the redistribution intensity parameter ϕ . I discuss this in the following remark.

Remark 2. [Winners and Losers of Redistribution] In equilibrium, redistribution benefits only those players with sufficient contest strength. Inactive players ($i \notin \mathcal{L}$) always lose, as they forfeit a fraction ϕ of their wealth without receiving any return; i.e., $U_i^ = (1 - \phi)w_i < w_i$. Active players ($i \in \mathcal{L}$) are worse off when their redistribution share falls short of their initial contribution to the total redistributive surplus (condition 2.7). However, for any given set of contest strengths, when the exogenous redistribution prize is sufficiently large, any active player can gain from redistribution.*

$$s_i = \left(\frac{c_i}{C}\right)^2 < \frac{\phi w_i}{\phi S_W + V} \quad (2.7)$$

2.2 Non-Monotonic Effects of Efficiency

In the central analysis of the paper, I now study how different types of contest efficiency gains affect redistribution outcomes and aggregate dissipation. I focus on two equilibrium objects, i.e., redistribution shares s_i and the dissipation factor δ . I then show that different type of efficiency gains could affect both of these objects in a non-monotonic way.

Importantly, the non-monotonicity does not hinge on the convexity of redistribution shares in relative strength (a feature pinned down by the chosen contest success function).

⁸The aggregate post-contest wealth is defined net of all effort costs.

Rather, it is generated by two forces that arise broadly across contest models:

- (i) how policy shifts the inefficiency cutoff for participation, potentially inducing entry or exit;
- (ii) how the resulting change in the cutoff—and any induced composition change—alters participants’ relative contest strengths.

The signs of the comparative-static effects then follow from (i)–(ii). (Convexity, when present, *amplifies magnitudes* of these effects but does not determine their direction.) As discussed in Section 2.3, the economic mechanism is therefore robust to the specification of the contest success function.

2.2.1 Redistribution shares

I consider three types of exogenous efficiency gains: (i) targeted efficiency gains for inactive players (without inducing entry) as well as for active players (targeted reform without inclusion); (ii) entry of previously inactive players into the active set, driven by exogenous efficiency gains (inclusion-oriented reform); and (iii) uniform improvements in efficiency that apply to all players (uniform reform). Propositions 2–4 characterize how targeted, inclusion-oriented, and uniform efficiency reforms affect post-contest wealth. Table 1 summarizes the direction of these effects for different reform types.

Table 1: Direction of Redistribution-Share Changes under Different Efficiency Gain

Efficiency Gain Type	Inactive	Weak Active	Strong Active
(1) Targeted gain to inactives (extensive margin)	–	–	–
(2) Targeted gain to weak actives (intensive margin)	–	↑	↓
(3) Entry of inactive (participation-expanding reform)	↑	↓	↓
(4) Equal efficiency gain (uniform reform)	–	↓	↑

The results show that, among all forms of efficiency gains, only inclusion-oriented reforms—those that bring previously inactive players into the competition—narrow the wealth gap between active and inactive participants without increasing the redistribution shares of already active players. By contrast, a uniform policy that improves the efficiency of all players neither enables inactive players to enter nor narrows disparities within the active set; instead, it widens the wealth gap among active participants. Consequently, uniform efficiency gains primarily benefit the already efficient and exacerbate inequality within the active set.

I begin with the comparative statics of redistribution shares with respect to individual efficiency. I first consider the case in which no player is too close to the participation threshold, i.e., there exists a sufficiently small ε such that $|\tilde{a} - a_i| > \varepsilon$ for all $i \in \mathcal{N}$.

Proposition 2 (Comparative Statics of redistribution shares with Respect to Inefficiency).

In the equilibrium characterized by Proposition 1,

(i) (Inactive players are irrelevant off-margin) For any inactive player $j \notin \mathcal{L}$, marginal changes in inefficiency do not affect the equilibrium redistribution shares:

$$\frac{\partial s_i}{\partial a_j} = 0 \quad \text{for all } i \in \mathcal{N}$$

(ii) (Own efficiency matters) For any active player $i \in \mathcal{L}$, an increase in own inefficiency reduces the redistribution share:

$$\frac{\partial s_i}{\partial a_i} < 0$$

(iii) (Relative strength governs redistribution) For any distinct pair $i, j \in \mathcal{L}$, if player j becomes less efficient, player i 's share rises:

$$\frac{\partial s_i}{\partial a_j} > 0$$

The formal proof is presented in Appendix 4. Part (i) follows directly from the participation threshold condition and the characterization of redistribution shares. When $a_j > \tilde{a}$, player j is inactive ($j \notin \mathcal{L}$) and exerts zero effort in equilibrium. A marginal change in a_j does not induce entry, as I consider here the non-generic case in which $\tilde{a} \neq a_i$ for all $i \in \mathcal{N}$; thus, the threshold $\tilde{a}$ remains unchanged. Furthermore, redistribution shares depend only on the distribution of inefficiencies within the active set and on the threshold $\tilde{a}$. Therefore, redistribution shares remain unaffected by marginal changes in off-margin players.

For part (ii), a marginal increase ε in the efficiency of an active player $i \in \mathcal{L}$ has two effects. First, the direct effect is to increase her contest strength, $c_i = \tilde{a} - a_i$, by ε . Second, the participation threshold $\tilde{a}$ declines by $\frac{\varepsilon}{L-1}$, capturing the indirect effect of increased competition.⁹ The net change in c_i is $\frac{(L-2)\varepsilon}{L-1}$. Thus c_i is unchanged when $L = 2$ and increases when $L > 2$. However, *relative* strength is $c_i/\mathcal{C}$, and since $\mathcal{C}$ falls by the same threshold shift, $c_i/\mathcal{C}$ rises for all $L \geq 2$.¹⁰ For other active players $j \neq i$, the decrease in $\tilde{a}$ reduces their contest strength, $c_j = \tilde{a} - a_j$, by $\frac{\varepsilon}{L-1}$. This implies that the total strength of the other active players decreases by ε . Including player i , the total contest strength of the active set changes by

$$\Delta \mathcal{C} = \frac{(L-2)\varepsilon}{L-1} - \varepsilon = \frac{-\varepsilon}{L-1} < 0$$

⁹Note that I assumed that there is no player $j \in \mathcal{L}$ such that $\tilde{a} - \varepsilon < a_j < \tilde{a}$ for sufficiently small ε . Thus, the active set remains unchanged.

¹⁰Intuitively, the indirect effect diminishes as the number of competing players increases.

Consequently, player i 's redistribution share, $s_i = \left(\frac{c_i}{\mathcal{C}}\right)^2$, increases.¹¹

For part (iii), suppose another active player $j \neq i$ becomes slightly more efficient by ε . This lowers the threshold by $\frac{\varepsilon}{L-1}$, i.e. $\tilde{a}' = \tilde{a} - \frac{\varepsilon}{L-1}$. Player i 's contest strength $c_i = \tilde{a} - a_i$ therefore falls by the same amount, $c'_i = c_i - \frac{\varepsilon}{L-1}$. By the equilibrium identity $\mathcal{C} = \tilde{a}$ (Proposition 1), the total strength decreases by the same amount as the threshold: $\mathcal{C}' = \mathcal{C} - \frac{\varepsilon}{L-1}$. Since both c_i and $\mathcal{C}$ drop by the same absolute amount and $c_i < \mathcal{C}$, the ratio $c_i/\mathcal{C}$ becomes smaller. Hence player i 's redistribution share, $s_i = \left(c_i/\mathcal{C}\right)^2$, decreases.

Therefore, relative contest strength among active players drives redistribution outcomes. When an inactive player becomes more efficient but still does not enter, relative contest strength of actives remains unchanged, and so does redistribution. But any efficiency gain by an already active player raises her redistribution share and reduces those of other actives.

The next proposition considers the case where an efficiency gain induces the entry of an player who was previously inactive.

Proposition 3 (Entry of a Previously Inactive player). *Let player $i \notin \mathcal{L}$ be initially inactive. Consider any reduction in her inefficiency, $a_i \mapsto a_i - \varepsilon$ that makes player i active and denote the share of any player $k \in N$ in the equilibrium as s'_k . Then:*

(i) *The entrant gets a strictly positive share of the contested surplus which is strictly increasing in ε :*

$$s'_i > 0$$

(ii) *(Every incumbent's share strictly declines) $\Delta s_j := s'_j - s_j < 0$ for all $j \in \mathcal{L}$.*

(iii) *(Inequality among incumbents widens) for any $j, k \in \mathcal{L}$ with $c_j > c_k$, the relative share ratio increases:*

$$\frac{s'_j}{s'_k} > \frac{s_j}{s_k}$$

The formal proof is in the Appendix. Let $A := \sum_{j \in \mathcal{L}} a_j$ denote the total inefficiency of the original active set, so the threshold is $\tilde{a} = \frac{A}{L-1}$. Without loss of generality, let the prospective entrant i satisfy $a_i^0 = \tilde{a}$, and suppose her inefficiency after the gain becomes $a_i^1 = \tilde{a} - \varepsilon$. The new threshold is $\tilde{a}' = \frac{A+a_i^1}{L}$. The total change in contest strength, $\Delta \mathcal{C}$, consists of two parts: a reduction in the strength of already active players (the indirect effect), and an increase due to the entrant's contest strength $c_i = \tilde{a}' - a_i^1$ (the direct effect).

¹¹Note that even in the special case $L = 2$, although there is no change in player i 's contest strength, the reduction in her rival's strength—and thus the total contest strength—still leads to an increase in player i 's share.

The aggregate reduction in already-active players' strength (the indirect effect) is:¹²

$$L(\tilde{a}' - \tilde{a}) = \underbrace{L \left(\frac{A + a_i^1}{L} - \frac{A}{L-1} \right)}_{\text{Indirect effect on already-active players}} = a_i^1 - \tilde{a} = -\varepsilon.$$

Also, the entrant's contest strength (the direct effect) is:

$$\underbrace{\tilde{a}' - a_i^1}_{\text{Direct effect of entrant}} = \frac{L-1}{L} \varepsilon = \varepsilon \left(1 - \frac{1}{L} \right).$$

Therefore, the net change in total contest strength is:

$$\Delta \mathcal{C} = \underbrace{-\varepsilon}_{\text{Indirect effect}} + \underbrace{\varepsilon \left(1 - \frac{1}{L} \right)}_{\text{Direct effect}} = -\frac{\varepsilon}{L} < 0.$$

Entry lowers total contest strength by ε/L .¹³ For any already-active player, the reduction in contest strength due to entry is the same, i.e., $\tilde{a} - \tilde{a}' = \frac{\varepsilon}{L}$. These imply that the shares of all previously active players decline (part (ii)). The new entrant also secures a strictly positive share proportional to her contest strength $c_i = \varepsilon(1 - 1/L) > 0$ (part (i)). Finally, because the absolute reduction in contest strength is the same for all already-active players, weaker players are affected more in relative terms—i.e., their shares decline proportionally more than those of more efficient players with higher initial contest strengths (part (iii)).

Thus, there are two changes due to inactive players entering the competition. First, inequality between active and inactive players decreases—the entrant obtains a positive share while the shares of already-active players decline. Second, while the redistribution shares of all already-active players decline in levels, the *ratio* of a highly efficient incumbent's share to that of a marginally active incumbent increases.

Importantly, the convexity of redistribution shares reinforces—but does not drive—these effects. Under a lottery contest success function, shares change quadratically with relative contest strength, and more generally remain convex when the return to effort in the CSF is sufficiently high (see Section 2.3). Hence, under convexity, an efficiency gain by the entrant magnifies the reduction in the shares of already active players, though the direction of change does not depend on convexity itself.

Intuitively, when returns to effort are high—as is often the case in low-capacity states

¹²Here, “direct” and “indirect” refer to the decomposition of the change in total contest strength—the entrant's own contribution versus the aggregate loss among incumbents—rather than to components of a single player's Δc_k as in Section 2.2.

¹³This occurs because the aggregate loss of strength by incumbents ($-\varepsilon$) is only partially offset by the entrant's new strength ($\varepsilon(1 - 1/L)$).

where access to resources depends heavily on individual initiative—an inclusion-oriented reform tends to have a stronger equalizing effect, reducing the wealth gap between inactive and active agents. In practice, however, such targeted interventions might not be feasible. Instead, to facilitate access to public aid, policymakers more commonly implement system-wide measures, such as digitization initiatives, institutional upgrades, or broad capacity-building programs.

The next result shows that even symmetric interventions that uniformly lower the cost of effort, i.e., lower each a_i by ε , can lead to unequal redistribution outcomes.

Proposition 4 (Redistribution Shares under Uniform Efficiency Gains). *Consider a uniform reduction in inefficiencies such that each a_i decreases by $\varepsilon > 0$. Then no inactive agents become active and for each $i \in \mathcal{L}$,*

$$\frac{\partial s_i}{\partial \varepsilon} = \begin{cases} > 0 & \text{if } c_i > \bar{c}_{\mathcal{L}} \\ < 0 & \text{if } c_i < \bar{c}_{\mathcal{L}} \\ = 0 & \text{if } c_i = \bar{c}_{\mathcal{L}} \end{cases}$$

where $\bar{c}_{\mathcal{L}}$ is the average contest strength among active players.

The formal proof is provided in Appendix 4. A uniform efficiency improvement lowers the participation threshold $\tilde{a}$ by $\frac{L\varepsilon}{L-1}$, which has two implications.¹⁴

First, no inactive player becomes active, since $\frac{L\varepsilon}{L-1} > \varepsilon$, i.e., the efficiency required for entry rises by more than each inactive player's own gain. Second, for each active player $k \in \mathcal{L}$, the reform has two opposing effects on contest strength: a direct increase of $+\varepsilon$ from the efficiency gain and an indirect decrease of $\frac{L\varepsilon}{L-1}$ due to the lower inefficiency threshold for participation. Together, these yield a net change of $-\frac{\varepsilon}{L-1}$ in each active player's contest strength, and a total decrease of $\frac{L\varepsilon}{L-1}$ for the active set as a whole.

Consider the redistribution shares before and after the uniform efficiency improvement:

$$\underbrace{s_i = \left(\frac{c_i}{\bar{c}}\right)^2}_{\text{before reform}}, \quad \underbrace{s'_i = \left(\frac{c_i - \frac{\varepsilon}{L-1}}{\bar{c} - \frac{L\varepsilon}{L-1}}\right)^2}_{\text{after reform}}$$

Following the reform, both individual contest strengths and the aggregate contest strength decline. Each active agent's contest strength decreases by the same absolute amount, $\varepsilon/(L-1)$, implying that the total contest strength falls by $L \times \varepsilon/(L-1)$. Because this aggregate loss scales with group size, agents who were initially stronger than average experience a smaller proportional decline and thus see their redistribution shares rise,

¹⁴To rule out incumbent exit after the uniform reform, assume $a_i < \tilde{a} - \frac{\varepsilon}{L-1}$ for every $i \in \mathcal{L}$. Indeed, under a uniform gain I have $a'_i = a_i - \varepsilon$ and $\tilde{a}' = \tilde{a} - \frac{L\varepsilon}{L-1}$; the no-exit requirement $a'_i < \tilde{a}'$ is equivalent to $a_i < \tilde{a} - \frac{\varepsilon}{L-1}$.

while weaker agents lose relatively more and their shares fall. Uniform efficiency gains therefore reallocate rewards toward the most efficient participants. Importantly, this effect does not rely on the convexity of contest shares with respect to relative strength, though convexity amplifies its magnitude.

2.2.2 Aggregate dissipation

This section examines how the aforementioned efficiency reforms affect aggregate dissipation. The dissipation factor δ , defined in Corollary 1, captures the degree of concentration in contest strengths within the active set. When contest strengths become more evenly distributed among active players, δ increases, reflecting higher aggregate dissipation.

I again consider three types of efficiency gains: (i) targeted improvements in efficiency for inactive players (without inducing entry) as well as for active players; (ii) inclusion-oriented reforms that bring previously inactive players into the active set through exogenous efficiency gains; and (iii) uniform improvements in efficiency that apply equally to all players (uniform reform).

Table 2 summarizes the results of this section. Targeted improvements in the efficiency of active players with low contest strength, which make the balance of strengths within the active set more even, increase aggregate dissipation. In contrast, when a strong active player becomes even stronger, concentration increases and aggregate dissipation decline.

Moreover, while small efficiency gains for inactive agents have no effect on aggregate dissipation, inducing entry does. Interestingly, if the entrant's contest strength becomes *sufficiently high*, aggregate dissipation falls. Together with Proposition 3, this suggests that a participation-expanding intervention that enables a previously inactive agent to compete 'effectively' can reduce both inequality and aggregate dissipation. Finally, although a uniform efficiency gain increases inequality, it reduces aggregate dissipation whenever contest strengths are not perfectly homogeneous. Thus, policymakers face a fundamental trade-off between equality and efficiency in aggregate dissipation when implementing uniform efficiency reforms.

Table 2: Direction of aggregate dissipation change under different types of efficiency interventions.

Reform Type	Feasible Change in $\mathcal{D}$	Condition for $\mathcal{D} \downarrow$
(i) Targeted gain to inactives (extensive margin)	0	N/A
(ii) Targeted gain to actives (intensive margin)	$\downarrow$ or $\uparrow$	Targeted active player is <i>strong</i>
(iii) Entry of inactives (participation-expanding)	$\downarrow$ or $\uparrow$	Entrants become sufficiently efficient
(iv) Equal efficiency gains (uniform reform)	$\downarrow$	Strict, unless $a_k = a_\ell$ for all $k, \ell \in \mathcal{L}$

Proposition 5 is the comparative statics of aggregate dissipation with respect to

inefficiencies, characterizing how changes in an individual player's inefficiency affect aggregate dissipation. I hold participation fixed here by assuming $a_i \neq \tilde{a}$ for all $i \in \mathcal{N}$.

Proposition 5 (Individual Efficiency and aggregate dissipation). *Suppose $a_i \neq \tilde{a}$ for all $i \in \mathcal{N}$. Then:*

- (i) (*extensive margin efficiency gain*) *For any player $j \notin \mathcal{L}$, changes in inefficiency have no effect on aggregate dissipation:*

$$\frac{\partial \mathcal{D}}{\partial a_j} = 0$$

- (ii) (*intensive-margin efficiency gain*) *For any player $j \in \mathcal{L}$, the direction of the effect depends on relative contest strength. Specifically:*

$$\frac{\partial \mathcal{D}}{\partial a_j} > 0 \quad \text{iff} \quad \frac{c_j}{\mathcal{C}} > \frac{\delta}{L-1} \quad (2.8)$$

Proof is available in Appendix 4. Part (i) states that if player $j \notin \mathcal{L}$, then a marginal change in her inefficiency a_j leaves equilibrium aggregate dissipation unchanged. The assumption $a_i \neq \tilde{a}$ for all $i \in \mathcal{N}$ rules out marginal cases where infinitesimal changes in inefficiency could induce entry, ensuring that the active set $\mathcal{L}$ remains fixed. As a result, a marginal change in the inefficiency of an inactive player a_j does not affect either the participation threshold $\tilde{a}$ or the contest strengths of active players (part (i)). This follows directly from the definition of $\tilde{a}$ in Lemma 1. So, part (i) says that no extensive-margin efficiency-enhancing intervention is effective.

When $j \in \mathcal{L}$, the effect of a change in a_j on aggregate dissipation depends on the player's relative position within the active set (part (ii)). A decrease in a_j has two conceptually distinct components. First, the *net own-effect (after accounting for the induced change in $\tilde{a}$)* on player j 's contest strength is

$$\Delta c_j = \frac{(L-2)\varepsilon}{L-1},$$

which raises her equilibrium effort.¹⁵ Second, there is a *system-wide effect* on the other incumbents: because $\tilde{a}$ falls by $\varepsilon/(L-1)$, every $k \neq j$ experiences

$$\Delta c_k = -\frac{\varepsilon}{L-1}.$$

¹⁵If I separate the pure "own" move from the feedback through the cutoff, then $c_j = \tilde{a} - a_j$ gives $\Delta c_j = (\Delta \tilde{a}) - (\Delta a_j) = (-\frac{\varepsilon}{L-1}) - (-\varepsilon) = \frac{(L-2)\varepsilon}{L-1}$. Thus the stated Δc_j already includes the threshold feedback.

The net impact on aggregate dissipation then depends on j 's initial strength: when j is relatively strong (large c_j/C), the reform reduces dissipation; when j is weak, it raises dissipation.

More specifically, intensive-margin interventions reduce aggregate dissipation if and only if the targeted active player's relative contest strength exceeds a threshold determined by the pre-policy dissipation factor and the size of the active set (condition 2.8). When aggregate dissipation is already high—that is, when δ is large—only a few very strong active players satisfy this condition, meaning that efficiency improvements for most actives will no longer reduce aggregate dissipation. Therefore, the higher the initial aggregate dissipation, the narrower the set of active players for whom targeted efficiency gains are effective in lowering aggregate dissipation.

Remark 3. While the extensive-margin interventions are not effective, intensive-margin policies entail trade-offs. Enhancing the efficiency of weaker participants reduces inequality but increases aggregate dissipation, whereas improving the efficiency of stronger participants lowers aggregate dissipation but widens inequality.

Proposition 6 shows that participation-expanding interventions—i.e., inclusion-oriented interventions—can reduce strategic aggregate dissipation, especially when the entrant is sufficiently efficient. Without loss of generality, I consider here a player whose inefficiency is initially at the participation threshold, $a_j = \tilde{a}$, who receives an efficiency gain $\varepsilon > 0$ (so that $a'_j := a_j - \varepsilon$).

Proposition 6 (Effective Entry Reduces aggregate dissipation). *Suppose the inefficiency of an inactive player j —whose inefficiency is initially at the participation threshold $a_j = \tilde{a}$ —is reduced by $\varepsilon > 0$. Then:*

- (i) *For any $\varepsilon > 0$, player j becomes active.*
- (ii) *Post-entry aggregate dissipation is lower than pre-entry aggregate dissipation if and only if*

$$\varepsilon > A \bar{c}_{\mathcal{L}} \delta \tag{2.9}$$

where δ denotes the dissipation factor before entry, $\bar{c}_{\mathcal{L}}$ is the average of contest strength of actives before entry, and $A := \frac{2}{\frac{L-1}{L} + \frac{\delta}{L^2}}$. When condition 2.9 holds, aggregate dissipation $\mathcal{D}$ decreases monotonically with respect to the entrant's efficiency gain ε .

Proof is provided in Appendix 4. There are a few forces at play here. Entry tends to increase aggregate dissipation, since an additional active player tend to intensify competition and increase total effort.

Entry also reshapes the distribution of contest strength by tightening the participation threshold, i.e., by lowering $\tilde{a}$. Specifically, the entry of player j increases the bar of competition; thereby uniformly reducing the contest strength of all already-active players. This reduction disproportionately weakens the weaker active players relative to the stronger ones. Moreover, the more efficient the entrant becomes, the wider the gap in relative strength among the already active players.

For example, consider an active set with two players whose contest strengths are 4 and 10 (a 10:4 ratio). If entry reduces each incumbent's contest strength by 1, the new ratio becomes 9:3. If the entrant is stronger so that the reduction equals 2, the ratio becomes 8:2. In the latter case, the relative-strength gap widens, further weakening the incentives of weaker active players to exert effort. This mitigating force could offset the competition effect and lowers overall aggregate dissipation, particularly when the entrant becomes sufficiently efficient.

The effect of entry on aggregate dissipation also depends on the pre-entry concentration of contest strength within the active set. To illustrate, let's hold ε fixed—i.e., keep the entrant's contest strength constant—and suppose the active set initially contains two players with contest strengths 1 and 10 (a 1:10 ratio). If entry reduces each active's strength by 0.5, the ratio of relative strengths after entry becomes $0.5:9.5 = 1:19$. However, if the initial strengths were 1 and 100 instead, the same reduction would yield $0.5:99.5 = 1:199$.

In the latter case, the gap in contest strength between the strong and weak actives widens further after entry. This, in turn, further weakens the weak active player's incentive to exert effort, leading to a greater reduction in aggregate dissipation. In general, when the initial heterogeneity in efficiency within the active set is sufficiently high, the disparity-increasing effect of entry can outweigh its competition-intensifying effect, leading to a net reduction in aggregate dissipation (Figure 3). This, in turn, makes participation-expanding interventions less dissipative when a few entrenched elites dominate the active set.

Precisely, the aforementioned economic intuitions are captured by condition 2.9. The entrant's efficiency gain should exceed a threshold given by the right-hand side of condition 2.9. Regarding the right-hand side of condition 2.9, two insights follow. First, holding the entrant's strength fixed, entry can reduce aggregate dissipation when the average contest strength among actives (c_L) is sufficiently low. Second, aggregate dissipation also falls when pre-entry aggregate dissipation (δ) is sufficiently low—or, equivalently, when contest strength is highly concentrated among a few players.

Finally, the coefficient A captures the *size effect*—that is, how responsive aggregate dissipation is to the entry of a new participant. Formally, the coefficient

$$A(L) = \frac{2}{\frac{L-1}{L} + \frac{\delta}{L^2}}$$

is strictly decreasing in the number of actives L . A larger A (small L) means a larger

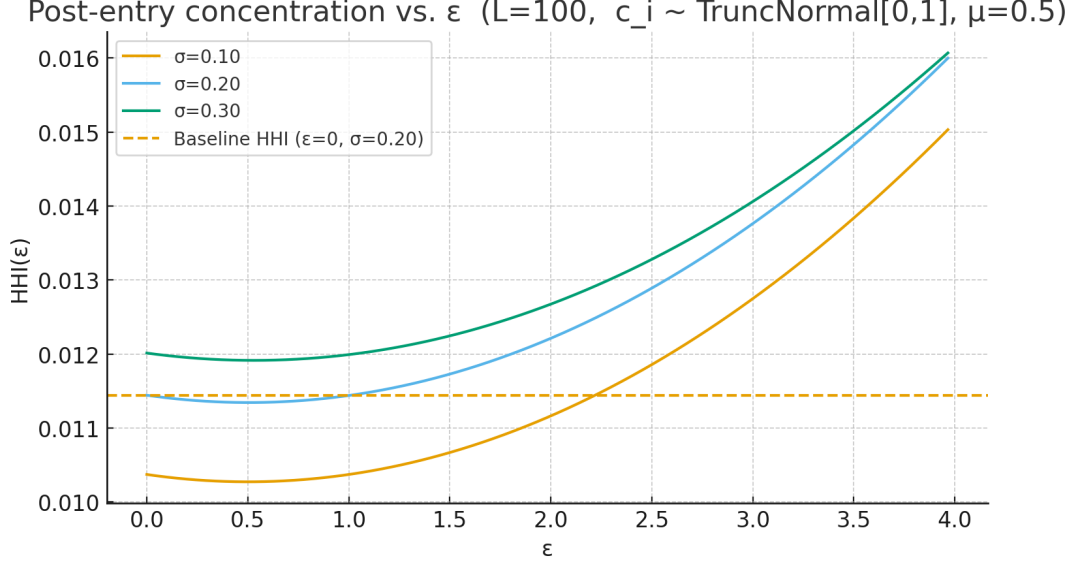


Figure 3: Post-entry concentration $HHI(\varepsilon)$ for $L = 100$ with $c_i \sim \text{TruncNormal}[0, 1](\mu = 0.5, \sigma)$. Solid curves: $\sigma \in \{0.10, 0.20, 0.30\}$. Dashed line: pre-entry $HHI(0)$ from the $\sigma = 0.20$ draw.

efficiency gain ε is required to reduce dissipation; entry is thus harder to make dissipation-reducing in small groups.

Remark 4. (No Equity-Efficiency Trade-off) Taken together with Proposition 3, these results imply that participation-expanding reforms that make entrants sufficiently efficient can reduce both aggregate dissipation and inequality between actives and inactives. This stands in contrast to the conventional view of an equity-efficiency trade-off. Importantly, such inclusion-oriented policies are most effective in settings where a few elites dominate the competition for redistribution.

Finally, I turn to uniform efficiency reforms. The next proposition shows that a uniform reduction in inefficiency among all active players unambiguously reduces aggregate dissipation, except in the case of perfect symmetry, where all active players are equally efficient.

Proposition 7 (Uniform Efficiency Gains Reduce aggregate dissipation). *Suppose that each active player experiences a uniform efficiency gain of magnitude $\varepsilon > 0$. Then:*

$$\frac{d\mathcal{D}}{d\varepsilon} \leq 0,$$

with equality if and only if the contest is perfectly symmetric—that is, $a_k = a_\ell$ for all $k, \ell \in \mathcal{L}$.

The formal proof is provided in Appendix 4. The underlying intuition is straightforward. As shown in Proposition 4, a uniform efficiency gain does not induce entry; rather, it either

leaves the active set unchanged or removes some weaker actives from competition by raising the bar for participation, i.e., by lowering the inefficiency threshold. For the remaining players, the uniform reform therefore reduces each participant's contest strength by the same absolute amount. Since these reductions apply uniformly, stronger participants—those with higher c_i —lose proportionally less in relative terms. Consequently, relative contest strength becomes more concentrated, thereby reducing aggregate dissipation.

Remark 5. While participation-expanding reforms could reduce both inequality and dissipation, uniform reforms inherently entail a trade-off between them, i.e., it increases inequality but lowers dissipation.

2.3 Robustness

The threshold-shifting mechanism is not tied to the linear-lottery benchmark. It arises in any contest with endogenous participation and heterogeneous efficiency where effort determines relative shares. Linear success functions and linear effort costs serve only to obtain closed-form expressions for post-redistribution wealth. These assumptions simplify exposition without changing the qualitative logic. More general contest technologies generate the same strategic forces, with convex redistribution shares merely amplifying them.

Uniform efficiency gains can paradoxically narrow, rather than broaden, participation. In contests where effort translates resources into redistributive advantage, equilibrium features a participation cutoff: only agents with inefficiency below this threshold exert positive effort. The cutoff adjusts so that the marginal participant is indifferent between entering and abstaining. When all costs fall uniformly, each agent raises effort to preserve relative standing, raising aggregate effort and intensifying competition. The resulting escalation shifts the indifference condition downward, tightening participation even as access costs fall equally for all.

Convex costs and strategic substitutability reinforce this tightening effect. When everyone becomes more efficient, all players expand effort, but efficient agents do so disproportionately because their lower base costs allow further escalation despite rising marginal costs. Their stronger response reweights the active set toward efficient players, lowering the cost-weighted mean inefficiency among participants. The equilibrium cutoff moves proportionally to this change: it falls whenever the average inefficiency of active players declines. As composition shifts toward greater efficiency, equilibrium requires a lower cutoff, leading to a more selective and unequal contest.

These forces extend beyond the benchmark specification. The generalized framework below introduces arbitrary concave success functions and convex effort costs and shows formally that the same threshold-shifting logic—and the associated equity–efficiency trade-off—persists under broad conditions.

2.3.1 Generalized framework

Consider a decentralized distribution with payoffs

$$U_i(e) = (1 - \phi)w_i + P p_i(e) - a_i h(e_i), \quad p_i(e) = \frac{f(e_i)}{\sum_k f(e_k)} \quad (2.10)$$

where $P = \phi S_W + V$ is the redistributive surplus, $p_i(e)$ is the share of player i , f, h are twice continuously differentiable, strictly increasing, and satisfy $f' > 0$, $f'' \leq 0$, $h' > 0$, $h'' \geq 0$, and the parameter $a_i > 0$ is player i 's inefficiency. The baseline specification is $f(e) = e$ and $h(e) = e$.¹⁶

2.3.2 Monotonicity and the participation cutoff

For any active player ($e_i > 0$), the first-order condition is

$$P \Xi_i(e) = a_i h'(e_i) \quad (2.11)$$

where $\Xi_i(e) = \frac{f'(e_i)}{\sum_k f(e_k)}(1 - \pi_i)$ is the marginal benefit from the effort, and $\pi_i = \frac{f(e_i)}{\sum_k f(e_k)}$ is the share of i from total effort. The right-hand side increases in e_i (since $h'' \geq 0$), while the left-hand side decreases in e_i (since $f'' \leq 0$ and the residual term $1 - \pi_i$ falls with e_i). Hence, for fixed e_{-i} and a_i , there is at most one interior solution. Moreover, $e_i^*(a_i; e_{-i}) > 0$ if and only if $P \Xi_i(0) > a_i h'(0)$, which defines a unique participation cutoff $\tilde{a}$ with $e_i^*(a_i; e_{-i}) > 0$ if and only if $a_i < \tilde{a}$, and e_i^* decreases continuously in a_i on $(0, \tilde{a})$.

2.3.3 Implicit characterization of the cutoff

Let $L(\tilde{a}) = \{i : a_i < \tilde{a}\}$ be the active set and $F = \sum_{k \in L} f(e_k)$. Summing (2.11) over $i \in L(\tilde{a})$ yields the aggregate interior FOCs

$$P \Xi(\tilde{a}) = \sum_{i \in L(\tilde{a})} a_i h'(e_i). \quad (2.12)$$

where $\Xi(\tilde{a}) := \sum_{i \in L(\tilde{a})} \frac{f'(e_i)}{F}(1 - \pi_i)$. Under standard concavity (e.g. Rosen's DSC), the active-set game admits a unique, continuously differentiable equilibrium $e_L^*(\tilde{a})$. Substituting $e_L^*(\tilde{a})$ into (2.12) gives

$$H_{\text{int}}(\tilde{a}) = P \sum_{i \in L(\tilde{a})} \frac{f'(e_i^*)}{F^*}(1 - \pi_i^*) - \sum_{i \in L(\tilde{a})} a_i h'(e_i^*) = 0, \quad (2.13)$$

which aggregates the interior FOCs; and for a fixed $L(\tilde{a})$, I have $H_{\text{int}}(\tilde{a}) \equiv 0$.

¹⁶I again abstract from individual liquidity constraints so that the cost side $a_i h(e_i)$ does not depend on w_i , i.e., effort is not directly constrained by own wealth.

Let denote $\partial_{e_i} \Xi_i(e^*) := P \frac{f'(e_i^*)}{F^*} (1 - \pi_i^*)$. The cutoff is determined by the boundary KKT condition for the marginal participant. Let j be such that $a_j = \tilde{a}$ and $e_j^* = 0$. Then, I have $P \partial_{e_j} \Xi_j(0^+, e_{-j}^*) = \tilde{a} h'(0)$. Hence,

$$\tilde{a} = \frac{P}{h'(0)} \partial_e \Xi(0^+, e_-^*(\tilde{a})), \quad (2.14)$$

Under monotone $\partial_e \Xi$ and single-crossing in a , the equilibrium boundary condition defines a one-dimensional fixed point. After substituting the equilibrium active-set mapping $L(\tilde{a}) = \{i : a_i < \tilde{a}\}$ and the corresponding equilibrium effort vector $e_-^*(\tilde{a})$ into (2.14), I obtain

$$G(\tilde{a}) := \frac{P}{h'(0)} \partial_e \Xi(0^+, e_-^*(\tilde{a}))$$

On any interval where $L(\tilde{a})$ is fixed, $e_-^*(\tilde{a})$ varies smoothly with $\tilde{a}$, and the total contest effort increases as less efficient agents enter, implying $\partial_e \Xi(0^+, e_-^*(\tilde{a}))$ —and hence $G(\tilde{a})$ —decrease strictly. At each inefficiency threshold where a new player enters, $G(\tilde{a})$ decreases by a discrete step. Thus $G(\tilde{a})$ is strictly decreasing (piecewise continuous with downward jumps), ensuring a unique solution to $\tilde{a} = G(\tilde{a})$.¹⁷

2.3.4 Uniform efficiency reforms and threshold shifts

Consider a uniform efficiency reform indexed by ε , with $da_i/d\varepsilon = -1$ for all i . Let the active set $L(\tilde{a})$ be locally fixed (no knife-edge agents). The interior equilibrium condition equating total marginal benefits and marginal costs can be written as

$$H_{\text{int}}(\tilde{a}, \varepsilon) = P \Xi(\tilde{a}, \varepsilon) - \sum_{i \in L(\tilde{a})} a_i h'(e_i^*) = 0 \quad (2.15)$$

In Appendix 4, I show that near a reference equilibrium, the aggregate term $\Xi(\tilde{a}, \varepsilon)$ admits the local approximation

$$\Xi(\tilde{a}, \varepsilon) \approx \frac{\tilde{C}_0}{S^*(L(\tilde{a}), \varepsilon)}, \quad S^*(L(\tilde{a}), \varepsilon) = \sum_{i \in L(\tilde{a})} e_i^*(\varepsilon)$$

where $\tilde{C}_0 > 0$ is a local scaling constant capturing the curvature of the contest success function. The inverse dependence on S^* reflects that as total effort increases, each player's marginal impact on the aggregate probability share Ξ declines proportionally: contest success depends on total intensity rather than its internal allocation.

¹⁷In the linear-lottery case $f(e) = e$, (2.14) reduces to the identity $\tilde{a} = \tilde{a}$ on any fixed- L interval; the global monotonicity comes from the endogenous change in $L(\tilde{a})$, recovering the closed-form $\tilde{a} = \frac{1}{L-1} \sum_{j \in L} a_j$.

Moreover, the marginal boundary condition for the cutoff player satisfies

$$H_{\text{bnd}}(\tilde{a}, \varepsilon) = \frac{P \tilde{C}_0}{S^*(\varepsilon)} - \tilde{a} = 0 \quad \Longleftrightarrow \quad \tilde{a} S^*(\varepsilon) = P C_0,$$

where $S^*(\varepsilon)$ denotes total equilibrium effort. This expresses the indifference of the marginal agent with inefficiency $a = \tilde{a}$: her cost times total contest intensity equals the effective prize constant. Differentiating with respect to ε gives

$$\frac{d\tilde{a}}{d\varepsilon} = - \frac{\tilde{a}}{S^*(\varepsilon)} \frac{dS^*}{d\varepsilon}$$

If a reform raises total effort ($dS^*/d\varepsilon > 0$), the effective prize term $P\tilde{C}_0/S^*$ falls and the cutoff decreases ($d\tilde{a}/d\varepsilon < 0$): competition tightens and entry becomes harder. If total effort falls, the cutoff relaxes. So a more-than-one-for-one cutoff response (one of the main forces deriving the results) occurs when

$$\left| \frac{d\tilde{a}}{d\varepsilon} \right| > 1 \quad \Longleftrightarrow \quad \frac{\tilde{a}}{S^*} \left| \frac{dS^*}{d\varepsilon} \right| > 1 \quad (2.16)$$

That is, a more-than-one-for-one cutoff response arises when the baseline contest is already *tight*—meaning that participation is highly selective, with the marginal inefficiency $\tilde{a}$ large relative to aggregate effort ($\tilde{a}/S^*$)—or when aggregate effort is *highly elastic* to the reform parameter ($|dS^*/d\varepsilon|$ is large). In such environments, even small shifts in the participation threshold can trigger disproportionately large changes in aggregate effort, as the equilibrium must restore the marginal indifference condition for the cutoff agent.

More rigorously, in Appendix 4, I show that the interior balance condition can be rewritten as

$$H_{\text{int}}(\tilde{a}, \varepsilon) \approx \frac{P \tilde{C}_0}{S^*(L(\tilde{a}), \varepsilon)} - \bar{a}_h = 0, \quad \bar{a}_h := \frac{\sum_{i \in L(\tilde{a})} a_i h'_i}{\sum_{i \in L(\tilde{a})} h'_i},$$

where $\bar{a}_h$ is the cost-weighted mean inefficiency among active players. Differentiating yields

$$\frac{d\bar{a}_h}{d\varepsilon} \approx - \frac{P \tilde{C}_0}{(S^*)^2} \frac{dS^*}{d\varepsilon}.$$

Combining both derivatives implies

$$\frac{d\tilde{a}}{d\varepsilon} = -\tilde{a} \frac{d\bar{a}_h}{d\varepsilon} \frac{S^*}{P \tilde{C}_0} = \frac{C_0}{\tilde{C}_0} \frac{d\bar{a}_h}{d\varepsilon},$$

where $\tilde{C}_0 = 2C_0$ as shown in Appendix 4, which obtains

$$\left| \frac{d\tilde{a}}{d\varepsilon} \right| = \frac{1}{2} \left| \frac{d\bar{a}_h}{d\varepsilon} \right| \quad (2.17)$$

Thus, a more than one-to-one cutoff shift, i.e., $\left| \frac{d\bar{a}}{d\varepsilon} \right| > 1$ implies a proportionally large change in the average cost composition of the active set.

Under convex costs ($h'' > 0$) and a fixed active set, differentiating the definition of $\bar{a}_h$ with respect to ε also yields

$$\frac{d\bar{a}_h}{d\varepsilon} \propto \sum_{i \in L(\bar{a})} (a_i - \bar{a}_h) h''(e_i^*) \frac{de_i^*}{d\varepsilon} \quad (2.18)$$

where the proportionality abstracts from a positive normalization factor $\sum_i h'(e_i^*)$. This expression follows directly from the quotient-rule differentiation of $\bar{a}_h = \frac{\sum_i a_i h'(e_i^*)}{\sum_i h'(e_i^*)}$, which captures how the cost weights $h'(e_i^*)$ shift as equilibrium efforts respond to the reform. Intuitively, $\bar{a}_h$ declines when efficient players ($a_i < \bar{a}_h$) expand effort more strongly, since convex costs ($h'' > 0$) assign greater influence to high-effort agents in the weighted average. The aggregate response of $\bar{a}_h$ therefore reflects the heterogeneity of individual effort adjustments.

According to Equation (2.17), a large negative $d\bar{a}/d\varepsilon$ arises when efficient participants ($a_i < \bar{a}_h$) increase effort sharply ($de_i^*/d\varepsilon > 0$), while less efficient players adjust little or reduce effort. When Condition (2.16) holds, a uniform efficiency reform widens the gap $a_i - \bar{a}_h$, making the right-hand side of Equation (.19) positive. To re-establish equilibrium, the most efficient agents—who carry the largest weights under convex costs—must expand effort further. This feedback magnifies inequality among active players, consistent with the amplification effect of uniform reforms derived in the benchmark model.¹⁸

2.4 Further Discussion: wealth–efficiency alignment

So far, I have shown how reforms affect the distribution of shares in the redistribution process. This perspective is most relevant when wealth inequality arises mainly from the redistribution shares itself—for instance, when pre-existing wealth differences are small compared to a large exogenous contested surplus, such as public funds or natural resources.

However, in settings where redistribution relies heavily on agents' wealth or where players can invest in efficiency, another key factor becomes important: the alignment between wealth and efficiency. In these cases, inequality depends not only on who participates, but also on whether the wealthy are the most efficient or whether efficiency is concentrated among the less wealthy. The mechanism described in Section 4 provides a basis for qualitatively discussing how uniform and participation-expanding reforms affect inequality and dissipation when wealth–efficiency alignment matters.

When wealth and efficiency are positively correlated, and individuals can use wealth

¹⁸Under standard single-crossing and strategic-substitute conditions, $de_i^*/d\varepsilon > 0$, and responsiveness declines with a_i .

to enhance efficiency, the core mechanism of this paper is reinforced: uniform reforms intensify inequality, while targeted inclusion remains the most effective corrective. For example, when wealth and efficiency move closely together, redistribution operates in a diverging-inequality regime. Wealthy agents are also efficient, entering with both higher endowments and greater ability to capture the surplus. Because contest shares increase with efficiency, post-contest wealth compounds initial advantages—the same agents who start ahead also secure larger redistributive gains. Uniform reforms, such as across-the-board reductions in access costs or bureaucratic frictions, amplify this imbalance by disproportionately benefiting already dominant players. The result is a further widening of inequality. In contrast, targeted policies that raise the efficiency or participation of weaker agents can offset this effect by broadening the active set and redistributing contest power. Even small-scale entry by near-threshold participants can slightly compress inequality, as their inclusion dilutes the dominance of incumbent elites.

In contrast, when wealth and efficiency are negatively correlated, redistribution operates in a *self-correcting regime*. Here, poorer agents are more efficient, and richer agents are less effective competitors. In such environments, redistribution naturally flows downward, as efficient but initially disadvantaged players capture surplus from inefficient elites. The same uniform reforms that exacerbate inequality under alignment now reduce it: efficiency gains disproportionately benefit the poorer, high-efficiency agents, compressing disparities. Conversely, entry by wealthy but inefficient players shifts surplus upward, potentially reversing equalization. Targeted support for efficient but low-wealth participants strengthens the equalizing effect, while assistance to inefficient incumbents undermines it.

These contrasting regimes highlight a key extension of the model: under substantial initial inequality, the redistributive impact of reform depends critically on the underlying *wealth–efficiency alignment*. When the rich are also efficient—and especially when they can convert wealth into efficiency—the threshold-shifting mechanism is reinforced: uniform reforms deepen inequality, and targeted or participation-expanding policies remain the only effective correctives. When the poor are more efficient, competition itself acts as a stabilizing force, making uniform reforms equalizing rather than divergent. In such settings, uniform reforms can simultaneously reduce inequality and aggregate dissipation.

Thus, the same policy can yield opposing outcomes depending on whether efficiency advantages reinforce or offset wealth disparities. This extends the model’s central insight: even when participation is held fixed, the correlation between wealth and efficiency determines whether redistribution amplifies or mitigates inequality. Recognizing this alignment is essential for designing reforms that expand opportunity without entrenching advantage.

Table 3 summarizes these insights. It shows how the effects of uniform and participation-

expanding reforms depend on whether the underlying wealth–efficiency alignment is positive (reinforcing) or negative (offsetting). Desirable policy configurations—those that simultaneously reduce both inequality and dissipation—are highlighted in blue.

Table 3: Policy Effects under Wealth–Efficiency Alignment Regimes

Policy Type	Positive Align.		Negative Align.		Effect on Mech. (Sec. 2.2)
	Ineq.	Diss.	Ineq.	Diss.	
Uniform Reform	↑	↓	↓	↓	Reinforcing
Participation-Expanding	↓	↓ (if entrant strong)	↓	↑ (if entrant weak)	Offsetting

3 Conclusion

This paper has developed a simple theoretical framework in which redistribution emerges from decentralized contestation over a common surplus. Players differ in their strategic efficiency and choose endogenously whether to incur costly effort to appropriate or defend wealth. As a result, participation is selective, and redistributive outcomes are shaped not only by formal policy parameters but also by the distribution of effort capacity and the structure of strategic interaction.

The analysis delivers two main results. First, uniform efficiency gains—such as across-the-board reductions in bureaucratic frictions or access costs—can backfire. By tightening the endogenous participation threshold, these reforms fail to bring new players into the contest and instead amplify the strategic advantage of already active, highly efficient agents. Inequality among beneficiaries rises even as aggregate dissipation falls, generating an endogenous equity–efficiency trade-off. Second, participation-expanding interventions that move marginal agents across the entry threshold can jointly improve equity and efficiency. By enlarging the active set and diluting concentrated contest power, such reforms reduce both inequality and dissipation, particularly when entrants are sufficiently efficient and pre-reform contest strength is highly concentrated.

References

- Acemoglu, D. and J. A. Robinson (2008). Persistence of power, elites, and institutions. *American Economic Review* 98(1), 267–293.
- Aghion, P., U. Akcigit, and P. Howitt (2015). The schumpeterian growth paradigm. *Annual review of economics* 7(1), 557–575.
- Aghion, P., A. Bergeaud, T. Boppart, P. J. Klenow, and H. Li (2023). A theory of falling growth and rising rents. *Review of Economic Studies* 90(6), 2675–2702.

- Aghion, P. and P. Bolton (1997). A theory of trickle-down growth and development. *The review of economic studies* 64(2), 151–172.
- Andrews, C., A. de Montesquiou, I. A. Sánchez, P. V. Dutta, B. V. Paul, S. Samaranayake, J. Heisey, T. Clay, and S. Chaudhary (2021). *The state of economic inclusion report 2021: The potential to scale*. World Bank Publications.
- Banerjee, A., E. Breza, E. Duflo, and C. Kinnan (2019). Can microfinance unlock a poverty trap for some entrepreneurs? Technical report, National Bureau of Economic Research.
- Banerjee, A., E. Duflo, R. Glennerster, and C. Kinnan (2015). The miracle of microfinance? evidence from a randomized evaluation. *American economic journal: Applied economics* 7(1), 22–53.
- Banerjee, A., E. Duflo, C. Imbert, S. Mathew, and R. Pande (2020). E-governance, accountability, and leakage in public programs: Experimental evidence from a financial management reform in india. *American Economic Journal: Applied Economics* 12(4), 39–72.
- Bardhan, P. and D. Mookherjee (2000). Capture and governance at local and national levels. *American economic review* 90(2), 135–139.
- Bardhan, P. and D. Mookherjee (2006). Pro-poor targeting and accountability of local governments in west bengal. *Journal of development Economics* 79(2), 303–327.
- Björkman, M. and J. Svensson (2009). Power to the people: evidence from a randomized field experiment on community-based monitoring in uganda. *The Quarterly Journal of Economics* 124(2), 735–769.
- Casey, K., R. Glennerster, and E. Miguel (2012). Reshaping institutions: Evidence on aid impacts using a preanalysis plan. *The Quarterly Journal of Economics* 127(4), 1755–1812.
- Daminato, C. and L. Pistaferri (2024). Returns heterogeneity and consumption inequality over the life cycle. Technical report, National Bureau of Economic Research.
- Dutta, P., R. Murgai, M. Ravallion, and D. Van de Walle (2014). *Right to work?: assessing India’s employment guarantee scheme in Bihar*. World Bank Publications.
- Esteban, J. and D. Ray (1999). Conflict and distribution. *Journal of Economic Theory* 87(2), 379–415.
- Esteban, J. and D. Ray (2001). Collective action and the group size paradox. *American political science review* 95(3), 663–672.

- Esteban, J. and D. Ray (2011). Linking conflict to inequality and polarization. *American Economic Review* 101(4), 1345–1374.
- Fonseca, J. and A. Matray (2024). Financial inclusion, economic development, and inequality: Evidence from brazil. *Journal of Financial Economics* 156, 103854.
- Gabaix, X., J.-M. Lasry, P.-L. Lions, and B. Moll (2016). The dynamics of inequality. *Econometrica* 84(6), 2071–2111.
- Gaillard, A., P. Wangner, C. Hellwig, and N. Werquin (2023). Consumption, wealth, and income inequality: A tale of tails. *Available at SSRN 4636704*.
- Galor, O. and O. Moav (2004). From physical to human capital accumulation: Inequality and the process of development. *The review of economic studies* 71(4), 1001–1026.
- Gradstein, M. (1995). Intensity of competition, entry and entry deterrence in rent seeking contests. *Economics & Politics* 7(1), 79–91.
- Impullitti, G. and P. Rendahl (2025). *Market power, growth, and wealth inequality*. Centre for Finance, Credit and Macroeconomics, School of Economics
- Konrad, K. A. (2009). *Strategy and dynamics in contests*. Oxford University Press.
- Moldovanu, B. and A. Sela (2001). The optimal allocation of prizes in contests. *American Economic Review* 91(3), 542–558.
- Moldovanu, B. and A. Sela (2006). Contest architecture. *Journal of Economic Theory* 126(1), 70–96.
- Muralidharan, K., P. Niehaus, and S. Sukhtankar (2016). Building state capacity: Evidence from biometric smartcards in india. *American Economic Review* 106(10), 2895–2929.
- Nti, K. O. (1999). Rent-seeking with asymmetric valuations. *Public Choice* 98(3), 415–430.
- Olken, B. A. (2007). Monitoring corruption: evidence from a field experiment in indonesia. *Journal of political Economy* 115(2), 200–249.
- Olken, B. A. and R. Pande (2012). Corruption in developing countries. *Annu. Rev. Econ.* 4(1), 479–509.
- Piketty, T. and E. Saez (2003). Income inequality in the united states, 1913–1998. *The Quarterly journal of economics* 118(1), 1–41.
- Reinikka, R. and J. Svensson (2004). Local capture: evidence from a central government transfer program in uganda. *The quarterly journal of economics* 119(2), 679–705.

Skaperdas, S. (1996). Contest success functions. *Economic theory* 7(2), 283–290.

Tu, Y., B. Chen, C. Liao, S. Wu, J. An, C. Lin, P. Gong, B. Chen, H. Wei, and B. Xu (2025). Inequality in infrastructure access and its association with health disparities. *Nature Human Behaviour*, 1–14.

Tullock, G. et al. (1980). Efficient rent seeking. *Toward a theory of the rent-seeking society* 97, 112.

4 Appendix

Proof of Proposition 1. Step 1 (Individual optimization). Each player $i \in \mathcal{N}$ chooses effort $e_i \geq 0$ to maximize utility in (2.4), taking $E_{-i} = \sum_{j \neq i} e_j$ as given, with total effort $E_N = e_i + E_{-i}$. The utility in Equation (2.2) can be written as

$$U_i(e_i; E_{-i}) = (1 - \phi)w_i + \frac{P e_i}{e_i + E_{-i}} - a_i e_i, \quad \text{where } P := \phi S_W + V$$

denotes the total contestable surplus.¹⁹

Player i 's problem is therefore

$$\max_{e_i \geq 0} \left\{ (1 - \phi)w_i + \frac{P e_i}{e_i + E_{-i}} - a_i e_i \right\}.$$

For $P > 0$, the objective is strictly concave since

$$\frac{\partial^2 U_i}{\partial e_i^2} = -\frac{2PE_{-i}}{(e_i + E_{-i})^3} < 0.$$

Hence, the first-order condition is sufficient for a unique optimum, and the KKT conditions imply

$$\frac{P(E_N - e_i^*)}{E_N^2} - a_i \leq 0, \quad e_i^* \geq 0, \quad e_i^* \left(\frac{P(E_N - e_i^*)}{E_N^2} - a_i \right) = 0.$$

Whenever $e_i^* > 0$, the first condition holds with equality:

$$\frac{P(E_N - e_i^*)}{E_N^2} = a_i.$$

Step 2 (Precluding trivial equilibria). Any pure-strategy Nash equilibrium must involve at least two active players.

¹⁹Equation (2.2) may appear different because $\phi S_R + V$ depends on effort through $S_R = S_W - \rho E_N$. Substituting $R_i = w_i - \rho e_i$ cancels the ρe_i terms, giving $U_i = (1 - \phi)w_i + \frac{e_i}{E_N}(\phi S_W + V) - (\kappa_i + \rho)e_i$, which is equivalent to the above expression.

Case 1: No active players. If all players choose $e_j = 0$, each receives P/N . A unilateral deviation by any i to a small effort $e_i = \varepsilon > 0$ yields

$$U_i(\varepsilon) = (1 - \phi)w_i + P - a_i\varepsilon > (1 - \phi)w_i + P/N$$

for sufficiently small ε . Thus, $E_N = 0$ cannot be an equilibrium.

Case 2: One active player. If only player i is active, $E_N = e_i$ and $U_i(e_i) = (1 - \phi)w_i + P - a_ie_i$. Since the prize P is independent of e_i , player i can profitably reduce effort, lowering cost without changing payoff. Hence, no equilibrium with a single active player exists.

Any equilibrium must therefore feature an active set $\mathcal{L} \subseteq \mathcal{N}$ with $L := |\mathcal{L}| \geq 2$.

Step 3 (Equilibrium effort and participation). For any $i \in \mathcal{L}$, the first-order condition with equality implies

$$P(E_N - e_i) = a_i E_N^2.$$

Summing over all active players gives

$$P(L - 1)E_N = E_N^2 \sum_{j \in \mathcal{L}} a_j \quad \Rightarrow \quad E_N^* = \frac{P(L - 1)}{\sum_{j \in \mathcal{L}} a_j}.$$

Define the participation threshold

$$\tilde{a} := \frac{1}{L - 1} \sum_{j \in \mathcal{L}} a_j, \quad \text{so that} \quad E_N^* = \frac{P}{\tilde{a}}.$$

Substituting into the first-order condition yields each active player's equilibrium effort:

$$e_i^* = \frac{P(\tilde{a} - a_i)}{\tilde{a}^2} > 0, \quad i \in \mathcal{L}, \quad e_j^* = 0, \quad j \notin \mathcal{L}.$$

Hence, player i is active iff $a_i < \tilde{a}$, so the active set is

$$\mathcal{L} = \{i \in \mathcal{N} : a_i < \tilde{a}\}.$$

Lemma 1 ensures that the participation threshold $\tilde{a}$ and active set $\mathcal{L}$ are unique. Given $\mathcal{L}$, strict concavity of U_i in e_i implies a unique equilibrium vector e^* .

Step 4 (Equilibrium utility). For any inactive player $j \notin \mathcal{L}$,

$$U_j^* = (1 - \phi)w_j.$$

For any active $i \in \mathcal{L}$, substituting the equilibrium effort into the utility function gives

$$U_i^* = (1 - \phi)w_i + P \left(\frac{e_i^*}{E_N^*} \right)^2,$$

where

$$\frac{e_i^*}{E_N^*} = \frac{\tilde{a} - a_i}{\tilde{a}} = \frac{c_i}{\tilde{a}}, \quad c_i := \max\{\tilde{a} - a_i, 0\}, \quad \mathcal{C} := \sum_{j \in \mathcal{L}} c_j = \tilde{a}.$$

Thus, the equilibrium redistribution share is

$$s_i := \left(\frac{c_i}{\mathcal{C}} \right)^2, \quad U_i^* = (1 - \phi)w_i + s_i(\phi S_W + V),$$

for all $i \in \mathcal{N}$. □

Proof of Corollary 1. Aggregate dissipation, denoted by $\mathcal{D}$, measures the total cost incurred by all active players in equilibrium. Let $\mathcal{L}$ be the set of active players, and let $P = \phi S_W + V$ denote the total contestable surplus. Then

$$\mathcal{D} = \sum_{i \in \mathcal{L}} a_i e_i^*.$$

From the first-order condition derived in Proposition 1, each active player satisfies $a_i = P \frac{E_N^* - e_i^*}{(E_N^*)^2}$. Substituting this into the expression above gives

$$\begin{aligned} \mathcal{D} &= \sum_{i \in \mathcal{L}} \left(P \frac{E_N^* - e_i^*}{(E_N^*)^2} \right) e_i^* = \frac{P}{(E_N^*)^2} \sum_{i \in \mathcal{L}} (E_N^* e_i^* - (e_i^*)^2) \\ &= \frac{P}{(E_N^*)^2} \left(E_N^* \sum_{i \in \mathcal{L}} e_i^* - \sum_{i \in \mathcal{L}} (e_i^*)^2 \right). \end{aligned}$$

Since total effort is $E_N^* = \sum_{i \in \mathcal{L}} e_i^*$, this simplifies to

$$\mathcal{D} = P \left(1 - \frac{\sum_{i \in \mathcal{L}} (e_i^*)^2}{(E_N^*)^2} \right).$$

The ratio $\sum_{i \in \mathcal{L}} (e_i^*)^2 / (E_N^*)^2$ represents the Herfindahl–Hirschman Index (HHI) of equilibrium effort shares, capturing how concentrated the contest is among active players. From Proposition 1, individual effort is proportional to each player's contest strength, $e_i^* = k c_i$ where $k = P/\tilde{a}^2$ and $c_i = \tilde{a} - a_i$. Substituting this proportionality yields

$$\frac{\sum_{i \in \mathcal{L}} (e_i^*)^2}{(E_N^*)^2} = \frac{\sum_{i \in \mathcal{L}} (k c_i)^2}{(\sum_{j \in \mathcal{L}} k c_j)^2} = \frac{\sum_{i \in \mathcal{L}} c_i^2}{(\sum_{j \in \mathcal{L}} c_j)^2} =: HHI(c).$$

Replacing this term in the expression for $\mathcal{D}$ gives a simple and intuitive form:

$$\mathcal{D} = P[1 - HHI(c)] = \delta(\phi S_W + V), \quad \text{where} \quad \delta := 1 - HHI(c).$$

Thus, aggregate dissipation is proportional to the contestable surplus, scaled by one minus the concentration of contest strength among the active players. $\square$

Proof of Proposition 2 (Comparative statics of redistribution shares). **(i) Inactive players.** Let $k \notin \mathcal{L}$. A marginal change in a_k does not affect the equilibrium provided it does not induce entry, i.e. as long as $a_k > \tilde{a}$. The participation threshold $\tilde{a}$, individual contest strengths c_i , and total contest strength $\mathcal{C}$ therefore remain unchanged.

It follows immediately that

$$\frac{\partial s_i}{\partial a_k} = 0 \quad \text{for all } i \in \mathcal{N}.$$

(ii) Own inefficiency. Let $i \in \mathcal{L}$. The derivative of $s_i = (c_i/\mathcal{C})^2$ with respect to a_i is

$$\frac{\partial s_i}{\partial a_i} = 2 \frac{c_i}{\mathcal{C}} \cdot \frac{(\partial c_i / \partial a_i) \mathcal{C} - c_i (\partial \mathcal{C} / \partial a_i)}{\mathcal{C}^2}.$$

Since $\tilde{a} = \frac{1}{L-1} \sum_{j \in \mathcal{L}} a_j$,

$$\frac{\partial \tilde{a}}{\partial a_i} = \frac{1}{L-1}, \quad \frac{\partial c_i}{\partial a_i} = \frac{\partial \tilde{a}}{\partial a_i} - 1 = -\frac{L-2}{L-1}, \quad \frac{\partial \mathcal{C}}{\partial a_i} = \frac{\partial \tilde{a}}{\partial a_i} = \frac{1}{L-1}.$$

Substituting these expressions gives

$$\frac{\partial s_i}{\partial a_i} = \frac{2c_i}{\mathcal{C}^3} \left[\left(-\frac{L-2}{L-1} \right) \mathcal{C} - c_i \left(\frac{1}{L-1} \right) \right] = -\frac{2c_i}{(L-1)\mathcal{C}^3} [(L-2)\mathcal{C} + c_i].$$

Because $L \geq 2$, $c_i > 0$, and $\mathcal{C} > 0$, the term in brackets is strictly positive. Hence $\frac{\partial s_i}{\partial a_i} < 0$: an increase in a player's inefficiency reduces her redistribution share.

(iii) Cross effects. Let $j \in \mathcal{L}$ with $j \neq i$. The derivative of s_i with respect to a_j is

$$\frac{\partial s_i}{\partial a_j} = \frac{2c_i}{\mathcal{C}^3} [(\partial c_i / \partial a_j) \mathcal{C} - c_i (\partial \mathcal{C} / \partial a_j)].$$

Here

$$\frac{\partial c_i}{\partial a_j} = \frac{\partial \tilde{a}}{\partial a_j} = \frac{1}{L-1}, \quad \frac{\partial \mathcal{C}}{\partial a_j} = \frac{1}{L-1}.$$

Substituting these gives

$$\frac{\partial s_i}{\partial a_j} = \frac{2c_i}{(L-1)\mathcal{C}^3} (\mathcal{C} - c_i).$$

Since $c_i < \mathcal{C}$ for any active i when $L \geq 2$, I have $(\mathcal{C} - c_i) > 0$, implying that $\frac{\partial s_i}{\partial a_j} > 0$. Thus,

improving another active player's efficiency (reducing a_j) lowers s_i , whereas increasing a_j benefits the others.

□

Proof of Proposition 3 (Entry of a previously inactive player). Let the initial state consist of an active set $\mathcal{L}$ of size L with threshold $\tilde{a}$. Pick a marginally inactive player i with $a_i = \tilde{a}$ (knife-edge)²⁰ and suppose her inefficiency falls to $a'_i = \tilde{a} - \varepsilon$ for some $\varepsilon > 0$. Assume ε is small enough that no incumbent exits, i.e.

$$\varepsilon < L \left(\tilde{a} - \max_{j \in \mathcal{L}} a_j \right),$$

so that $a_j < \tilde{a} - \varepsilon/L$ for all $j \in \mathcal{L}$ and the new active set is $\mathcal{L}' = \mathcal{L} \cup \{i\}$ of size $L + 1$.²¹

New threshold and total strength. The post-entry threshold is

$$\tilde{a}' = \frac{1}{L} \left(\sum_{j \in \mathcal{L}} a_j + a'_i \right) = \frac{1}{L} ((L-1)\tilde{a} + (\tilde{a} - \varepsilon)) = \tilde{a} - \frac{\varepsilon}{L}.$$

Since total contest strength equals the threshold in equilibrium, $\mathcal{C}' = \tilde{a}' = \tilde{a} - \varepsilon/L$.

(i) Entrant's share. The entrant's strength and share are

$$c'_i = \tilde{a}' - a'_i = \varepsilon \left(1 - \frac{1}{L} \right) > 0, \quad s'_i = \left(\frac{c'_i}{\mathcal{C}'} \right)^2 > 0,$$

both increasing in ε .

(ii) Incumbents' shares. For $j \in \mathcal{L}$,

$$c'_j = \tilde{a}' - a_j = c_j - \frac{\varepsilon}{L}, \quad s'_j = \left(\frac{c'_j}{\mathcal{C}'} \right)^2.$$

Hence

$$\Delta s_j = \left(\frac{c_j - \varepsilon/L}{\tilde{a} - \varepsilon/L} \right)^2 - \left(\frac{c_j}{\tilde{a}} \right)^2 = \frac{(c_j \tilde{a} - \frac{\varepsilon}{L}(\tilde{a} - c_j))^2 - (c_j \tilde{a})^2}{(\tilde{a} - \varepsilon/L)^2 \tilde{a}^2} < 0,$$

because $0 < c_j < \tilde{a}$ and $\varepsilon > 0$ imply the numerator is negative. Thus every incumbent's share strictly declines.

²⁰Equivalently, start with $a_i > \tilde{a}$ but arbitrarily close; the limiting argument is identical.

²¹This is the standard “generic separation / no knife-edge incumbent” condition used throughout (see the assumption preceding Proposition 2). It ensures local stability of the active set under small perturbations.

(iii) Inequality among incumbents. For $j, k \in \mathcal{L}$ with $c_j > c_k$,

$$\frac{s'_j}{s'_k} = \left(\frac{c_j - \varepsilon/L}{c_k - \varepsilon/L} \right)^2 \quad \text{is strictly increasing in } \varepsilon,$$

since the mapping $x \mapsto (x - \varepsilon/L)/(y - \varepsilon/L)$ is increasing in ε whenever $x > y > \varepsilon/L$. Therefore, although all incumbents' levels fall, the ratio of strong to weak shares rises: entry widens inequality among actives.

□

Proof of Proposition 4 (Redistribution shares under uniform efficiency gains). Consider a uniform reduction in inefficiency by $\varepsilon > 0$, which changes each player's parameter to $a'_k = a_k - \varepsilon$.

(i) Effect on participation. A player j enters the contest if their new inefficiency a'_j lies below the new participation threshold $\tilde{a}'$. Initially, for an inactive player $a_j \geq \tilde{a}$, I have $a'_j = a_j - \varepsilon$. The new threshold for an active set of size L is

$$\tilde{a}' = \frac{1}{L-1} \sum_{i \in \mathcal{L}} a'_i = \tilde{a} - \frac{L\varepsilon}{L-1}.$$

Entry requires

$$a_j - \varepsilon < \tilde{a} - \frac{L\varepsilon}{L-1} \quad \Longleftrightarrow \quad a_j - \tilde{a} < -\frac{\varepsilon}{L-1}.$$

Because any inactive player satisfies $a_j \geq \tilde{a}$, the inequality cannot hold. Hence, no inactive player enters following a small uniform gain.

Similarly, an active player $i \in \mathcal{L}$ exits if $a'_i \geq \tilde{a}'$, or equivalently

$$a_i - \tilde{a} \geq -\frac{\varepsilon}{L-1}.$$

Since $a_i - \tilde{a} < 0$ for all actives and the right-hand side approaches zero as $\varepsilon \rightarrow 0$, this condition is not met for sufficiently small ε . Therefore, under a sufficiently small uniform efficiency gain, no exits or entries occur, and the active set $\mathcal{L}$ of size $L \geq 2$ remains unchanged.

(ii) Effect on redistribution shares. Within the unchanged active set, each player's cost becomes $a'_i = a_i - \varepsilon$. The new contest strength for $i \in \mathcal{L}$ is

$$c'_i = \tilde{a}' - a'_i = \left(\tilde{a} - \frac{L\varepsilon}{L-1} \right) - (a_i - \varepsilon) = c_i - \frac{\varepsilon}{L-1},$$

and total contest strength is

$$\mathcal{C}' = \sum_{i \in \mathcal{L}} c'_i = \mathcal{C} - \frac{L\varepsilon}{L-1}.$$

To assess the impact on redistribution shares $s_i = (c_i/\mathcal{C})^2$, differentiate with respect to ε and evaluate at $\varepsilon = 0$:

$$\left. \frac{\partial s_i}{\partial \varepsilon} \right|_{\varepsilon=0} = \frac{2c_i}{\mathcal{C}^3} \left(\mathcal{C} \left. \frac{\partial c'_i}{\partial \varepsilon} \right|_{\varepsilon=0} - c_i \left. \frac{\partial \mathcal{C}'}{\partial \varepsilon} \right|_{\varepsilon=0} \right).$$

Since $\left. \frac{\partial c'_i}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{1}{L-1}$ and $\left. \frac{\partial \mathcal{C}'}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{L}{L-1}$, I obtain

$$\left. \frac{\partial s_i}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{2c_i}{(L-1)\mathcal{C}^3} (\mathcal{C} - Lc_i).$$

Because the average contest strength is $\bar{c} = \mathcal{C}/L$, the term in parentheses equals $L(\bar{c} - c_i)$, giving

$$\left. \frac{\partial s_i}{\partial \varepsilon} \right|_{\varepsilon=0} = -\frac{2Lc_i}{(L-1)\mathcal{C}^3} (\bar{c} - c_i).$$

An efficiency gain reduces all a_i , so a positive ε corresponds to a smaller cost parameter. The sign of the derivative therefore depends on $(c_i - \bar{c})$: players whose contest strength exceeds the average ($c_i > \bar{c}$) experience an increase in their redistribution share, while those below the average see their share fall.

Hence, a uniform efficiency gain leaves participation unchanged but amplifies inequality among active players' redistribution shares. $\square$

Proof of Proposition 5 (Individual efficiency and aggregate dissipation). The active set $\mathcal{L}$ is assumed to be fixed.

(i) Inactive players ($j \notin \mathcal{L}$). Aggregate dissipation is given by $\mathcal{D} = P(1 - HHI(c))$, where P is the contestable surplus and $HHI(c)$ is the Herfindahl–Hirschman Index of the contest strengths $\{c_i\}_{i \in \mathcal{L}}$. Because $\tilde{a}$, $\mathcal{L}$, and $\{c_i\}$ depend only on the inefficiencies of active players, a marginal change in the inefficiency a_j of an inactive player has no effect on these quantities. Hence $HHI(c)$ remains constant, and

$$\frac{\partial \mathcal{D}}{\partial a_j} = 0.$$

(ii) Active players ($j \in \mathcal{L}$). For an active player, the effect of a change in a_j follows from $\mathcal{D} = P(1 - HHI(c))$, so that $\frac{\partial \mathcal{D}}{\partial a_j} = -P \frac{\partial(HHI)}{\partial a_j}$. The HHI is defined as

$$HHI(c) = \frac{\sum_{i \in \mathcal{L}} c_i^2}{(\sum_{k \in \mathcal{L}} c_k)^2} = \frac{\sum_{i \in \mathcal{L}} c_i^2}{\mathcal{C}^2}.$$

Differentiating using the quotient rule gives

$$\frac{\partial(HHI)}{\partial a_j} = \frac{\left(\partial(\sum c_i^2)/\partial a_j\right)\mathcal{C}^2 - (\sum c_i^2)\left(\partial\mathcal{C}^2/\partial a_j\right)}{\mathcal{C}^4}.$$

Step 1: Derivative of the numerator.

$$\frac{\partial}{\partial a_j} \left(\sum_{i \in \mathcal{L}} c_i^2 \right) = 2c_j \frac{\partial c_j}{\partial a_j} + \sum_{i \in \mathcal{L}, i \neq j} 2c_i \frac{\partial c_i}{\partial a_j}.$$

From Proposition 2,

$$\frac{\partial c_j}{\partial a_j} = -\frac{L-2}{L-1}, \quad \frac{\partial c_i}{\partial a_j} = \frac{1}{L-1} \quad (i \neq j).$$

Substituting gives

$$\frac{\partial(\sum c_i^2)}{\partial a_j} = \frac{2}{L-1} \left(-(L-2)c_j + \sum_{i \neq j} c_i \right) = \frac{2}{L-1} (\mathcal{C} - (L-1)c_j).$$

Step 2: Derivative of the denominator.

$$\frac{\partial\mathcal{C}^2}{\partial a_j} = 2\mathcal{C} \frac{\partial\mathcal{C}}{\partial a_j} = \frac{2\mathcal{C}}{L-1},$$

since $\frac{\partial\mathcal{C}}{\partial a_j} = \frac{1}{L-1}$ (Proposition 2).

Step 3: Substituting these results into the quotient rule yields

$$\begin{aligned} \frac{\partial(HHI)}{\partial a_j} &= \frac{1}{\mathcal{C}^4} \left[\frac{2}{L-1} (\mathcal{C} - (L-1)c_j) \mathcal{C}^2 - (\sum c_i^2) \frac{2\mathcal{C}}{L-1} \right] \\ &= \frac{2}{(L-1)\mathcal{C}^3} \left[\mathcal{C}^2 - (L-1)c_j\mathcal{C} - \mathcal{C}^2 HHI \right]. \end{aligned}$$

Simplifying,

$$\frac{\partial(HHI)}{\partial a_j} = \frac{2}{(L-1)\mathcal{C}} \left[(1 - HHI) - (L-1) \frac{c_j}{\mathcal{C}} \right].$$

Let $\delta := 1 - HHI$ denote the dissipation factor. Then

$$\frac{\partial(HHI)}{\partial a_j} = \frac{2}{(L-1)\mathcal{C}} \left[\delta - (L-1) \frac{c_j}{\mathcal{C}} \right].$$

Step 4: Effect on dissipation. Since $\mathcal{D} = P(1 - HHI)$,

$$\frac{\partial\mathcal{D}}{\partial a_j} = -P \frac{\partial(HHI)}{\partial a_j} = \frac{2P}{(L-1)\mathcal{C}} \left[(L-1) \frac{c_j}{\mathcal{C}} - \delta \right].$$

Because P , L , and $\mathcal{C}$ are positive, the sign of $\frac{\partial \mathcal{D}}{\partial a_j}$ depends only on the term in brackets. Hence,

$$\frac{\partial \mathcal{D}}{\partial a_j} \geq 0 \quad \Longleftrightarrow \quad \frac{c_j}{\mathcal{C}} \geq \frac{\delta}{L-1}.$$

□

Proof of Proposition 6 (Effect of entry on aggregate dissipation). Let $\mathcal{L}$ denote the initial set of L active players with contest strengths $\{c_i\}_{i \in \mathcal{L}}$, and define

$$C := \sum_{i \in \mathcal{L}} c_i, \quad S_2 := \sum_{i \in \mathcal{L}} c_i^2.$$

I seek the condition on the efficiency gain $\varepsilon > 0$ such that aggregate dissipation decreases following the entry of a new player.

After entry, the new participant has contest strength $c_{L+1} = \frac{(L-1)\varepsilon}{L}$, while each incumbent's strength falls to $c'_i = c_i - \frac{\varepsilon}{L}$. Aggregate dissipation decreases if and only if the new Herfindahl–Hirschman Index (HHI) increases, i.e.

$$\frac{\left(\frac{(L-1)\varepsilon}{L}\right)^2 + \sum_{i \in \mathcal{L}} \left(c_i - \frac{\varepsilon}{L}\right)^2}{\left(\frac{(L-1)\varepsilon}{L} + \sum_{i \in \mathcal{L}} \left(c_i - \frac{\varepsilon}{L}\right)\right)^2} > \frac{S_2}{C^2}. \quad (4.1)$$

Expanding the numerator and denominator of the left-hand side gives

$$\text{Num} = \frac{(L^2 - L + 1)\varepsilon^2}{L^2} - \frac{2C\varepsilon}{L} + S_2, \quad \text{Den} = \left(C - \frac{\varepsilon}{L}\right)^2.$$

Since denominators are positive, cross-multiplication preserves the inequality in (4.1), leading to

$$\left(\frac{C^2(L^2 - L + 1) - S_2}{L^2}\right)\varepsilon^2 + \left(\frac{2CS_2 - 2C^3}{L}\right)\varepsilon > 0.$$

Factoring out ε gives

$$\varepsilon \left[\left(\frac{C^2(L^2 - L + 1) - S_2}{L^2}\right)\varepsilon - \left(\frac{2C}{L}\right)(C^2 - S_2) \right] > 0.$$

Because $C^2(L^2 - L + 1) - S_2 > 0$ (as $S_2 \leq C^2$ and $L^2 - L + 1 \geq 1$), the quadratic opens upward. For $\varepsilon > 0$, the inequality holds whenever

$$\varepsilon > \frac{2CL(C^2 - S_2)}{C^2(L^2 - L + 1) - S_2}.$$

Using $S_2 = C^2 HHI(0)$ and $\delta(0) := 1 - HHI(0)$, I can rewrite the condition as

$$\varepsilon > \frac{2CLC^2\delta(0)}{C^2(L^2 - L + 1 - HHI(0))} = \frac{2CL\delta(0)}{L^2 - L + \delta(0)} = \frac{2C\delta(0)}{L - 1 + \frac{\delta(0)}{L}}.$$

This gives the minimum efficiency gain required for entry to *reduce* aggregate dissipation.

Define

$$A(L) := \frac{2}{\frac{L-1}{L} + \frac{\delta}{L^2}} = \frac{2}{1 - \frac{1}{L} + \frac{\delta}{L^2}}.$$

Differentiating $A(L) = \frac{2}{1 - \frac{1}{L} + \frac{\delta}{L^2}}$ with respect to L holding the pre-entry dissipation factor δ fixed (i.e., a partial derivative) yields

$$\frac{\partial A}{\partial L} = -2 \frac{\frac{d}{dL}\left(1 - \frac{1}{L} + \frac{\delta}{L^2}\right)}{\left(1 - \frac{1}{L} + \frac{\delta}{L^2}\right)^2} = -\frac{2(L - 2\delta)}{L^3\left(1 - \frac{1}{L} + \frac{\delta}{L^2}\right)^2}.$$

Since $L > 2$ and $0 < \delta < 1$, the numerator is positive, so $\frac{\partial A}{\partial L} < 0$. Hence, keeping the baseline concentration δ constant, the threshold ε required for entry to reduce aggregate dissipation decreases as the incumbent set grows.

Proof of Proposition 7 (Uniform efficiency gains reduce aggregate dissipation). Consider a uniform efficiency gain of magnitude $\varepsilon > 0$, which reduces each active player's inefficiency from a_i to $a'_i = a_i - \varepsilon$. Since the active set $\mathcal{L}$ remains fixed for small ε , the post-reform contest strengths are

$$c'_i = \tilde{a}' - a'_i = \left(\tilde{a} - \frac{L\varepsilon}{L-1}\right) - (a_i - \varepsilon) = c_i - \frac{\varepsilon}{L-1},$$

and the total contest strength becomes

$$\mathcal{C}' = \sum_{i \in \mathcal{L}} c'_i = \mathcal{C} - \frac{L\varepsilon}{L-1}.$$

Aggregate dissipation is given by $\mathcal{D} = P(1 - HHI)$, where HHI denotes the Herfindahl–Hirschman Index of relative contest strengths:

$$HHI = \frac{\sum_{i \in \mathcal{L}} c_i^2}{\mathcal{C}^2}.$$

Differentiating with respect to ε and evaluating at $\varepsilon = 0$ gives

$$\left. \frac{d(HHI)}{d\varepsilon} \right|_{\varepsilon=0} = \frac{2}{\mathcal{C}(L-1)} (L HHI_{\mathcal{L}} - 1).$$

This follows from substituting $\frac{\partial c'_i}{\partial \varepsilon} = -\frac{1}{L-1}$ and $\frac{\partial \mathcal{C}'}{\partial \varepsilon} = -\frac{L}{L-1}$ into the derivative of $HHI =$

$\frac{\sum c_i^2}{C^2}$ and simplifying.

By the Cauchy–Schwarz inequality, $HHI_{\mathcal{L}} \geq \frac{1}{L}$, with equality only if all contest strengths are identical. Thus,

$$L \cdot HHI_{\mathcal{L}} - 1 \geq 0 \implies \frac{d(HHI)}{d\varepsilon} \geq 0.$$

A uniform efficiency gain therefore increases (or leaves unchanged) the concentration of contest strengths: the distribution of effective contest power becomes more unequal.

Since aggregate dissipation moves inversely with the HHI ,

$$\frac{d\mathcal{D}}{d\varepsilon} = -P \frac{d(HHI)}{d\varepsilon} = -\frac{2P}{C(L-1)} (L HHI_{\mathcal{L}} - 1) \leq 0.$$

Hence, uniform efficiency gains can never increase aggregate dissipation. The inequality is strict whenever contest strengths are heterogeneous ($HHI_{\mathcal{L}} > 1/L$), and becomes an equality only under perfect symmetry. Notice that such an improvements (holding the active set fixed) reduce the relative dispersion of cost parameters a_i but—through the induced movement of the cutoff $\tilde{a}$ —make the distribution of contest strengths $c_i := \tilde{a} - a_i$ more unequal, raising concentration (HHI) and thereby lowering aggregate dissipation $\delta = 1 - HHI(c)$.²²

□

Appendix A. Local Approximation of the Participation Cutoff

Consider the contest success function

$$\pi_i = \frac{f(e_i)}{F}, \quad F = \sum_{k \in L} f(e_k),$$

where $f' > 0$ and $f'' \leq 0$, and let the cost function h satisfy $h' > 0$ and $h'' \geq 0$. Each active player $i \in L$ satisfies the first-order condition

$$\frac{P f'(e_i)}{F} (1 - \pi_i) = a_i h'(e_i), \quad (.2)$$

²²To avoid confusion, the concentration index is computed over strengths $\{c_i\}_{i \in \mathcal{L}}$, not costs $\{a_i\}$. A uniform efficiency shift can compress the dispersion of a_i while $\tilde{a}$ co-moves so that each incumbent's c_i falls by the same absolute amount; this equal subtraction increases the concentration of $\{c_i\}$ (and thus lowers δ). In the symmetric benchmark ($HHI_{\mathcal{L}} = 1/L$), strengths remain equal after the common subtraction, so HHI—and therefore δ —are unchanged.

which equates marginal expected gain to marginal cost. Summing (.2) over $i \in L$ yields

$$P \sum_{i \in L} \frac{f'(e_i)}{F} (1 - \pi_i) = \sum_{i \in L} a_i h'(e_i). \quad (.3)$$

Let $e^*(\tilde{a})$ denote the (locally) unique equilibrium effort vector in the active set L and define

$$\Xi(\tilde{a}) := \sum_{i \in L} \frac{f'(e_i^*)}{F^*} (1 - \pi_i^*), \quad S^* := \sum_{i \in L} e_i^*, \quad F^* := \sum_{i \in L} f(e_i^*).$$

Then (.3) can be written as

$$H(\tilde{a}) := P \Xi(\tilde{a}) - \sum_{i \in L} a_i h'(e_i^*) = 0. \quad (.4)$$

Let us fix a reference equilibrium $(\tilde{a}_0, e_0^*)$ and define

$$S_0 := S^*(\tilde{a}_0), \quad F_0 := F^*(\tilde{a}_0), \quad \Xi_0 := \Xi(\tilde{a}_0).$$

A first-order expansion of $H(\tilde{a})$ around $\tilde{a}_0$ gives

$$0 \approx H(\tilde{a}_0) + P \left[\Xi(e^*) - \Xi(e_0^*) \right] - \sum_{i \in L} (a_i - a_{i,0}) h'_i - \sum_{i \in L} a_{i,0} (h'(e_i^*) - h'_i), \quad (.5)$$

where $h'_i := h'(e_{i,0}^*)$ and $H(\tilde{a}_0) = 0$ at the reference point. Define the local constant

$$C_0 := S_0 \Xi_0 = \sum_{i \in L} S_0 \frac{f'(e_{i,0}^*)}{F_0} (1 - \pi_{i,0}^*), \quad (.6)$$

which is the contest's total marginal reward capacity at the reference point.

From multi-dimensional to one-dimensional: Ξ as a function of S^* . Let $\Delta e := e^* - e_0^*$. Project Δe onto the gradient of Ξ at e_0^* :

$$\alpha := \frac{\nabla \Xi(e_0^*)^\top \Delta e}{\nabla \Xi(e_0^*)^\top e_0^*}, \quad \zeta := \Delta e - \alpha e_0^*.$$

By construction, $\nabla \Xi(e_0^*)^\top \zeta = 0$. A first-order Taylor expansion yields

$$\Xi(e^*) - \Xi(e_0^*) = \nabla \Xi(e_0^*)^\top \Delta e + O(\|\Delta e\|^2) = \alpha \nabla \Xi(e_0^*)^\top e_0^* + O(\|\Delta e\|^2).$$

Since $S^* - S_0 = \mathbf{1}^\top \Delta e = \alpha \mathbf{1}^\top e_0^* + \mathbf{1}^\top \zeta$ and $\mathbf{1}^\top \zeta = O(\|\zeta\|)$, I can write, to first order,

$$\Xi(e^*) - \Xi(e_0^*) = -\frac{C_0}{S_0^2} (S^* - S_0) + O(\|\Delta e\|^2),$$

which implies the local inverse-in- S^* representation

$$\Xi(\tilde{a}) = \frac{C_0}{S^*(\tilde{a})} + R_\Xi(\tilde{a}), \quad R_\Xi(\tilde{a}_0) = 0, \quad |R_\Xi(\tilde{a})| \leq K_\Xi \left| \frac{S^*(\tilde{a}) - S_0}{S_0} \right|^2. \quad (.7)$$

Intuition. The argument above formalizes the idea that near an equilibrium profile e_0^* , the marginal-benefit term $\Xi(e)$ behaves essentially as a one-dimensional function of the aggregate effort $S^* = \sum_i e_i^*$. The projection α isolates the component of the adjustment $\Delta e = e^* - e_0^*$ that is aligned with the direction of proportional scaling of all efforts, e_0^* , while ζ captures purely redistributive deviations that leave the weighted average $\nabla \Xi(e_0^*)^\top e_0^*$ unchanged to first order. Since Ξ depends on e only through relative effort proportions, perturbations orthogonal to e_0^* (i.e., changes in composition holding total effort fixed) affect Ξ only at second order. Hence, the leading-order change in Ξ comes solely from the aggregate expansion or contraction of total effort S^* . The Taylor expansion then implies that $\Xi(e)$ is locally proportional to $1/S^*$ (with curvature term $R_\Xi(\tilde{a})$ capturing higher-order deviations.) Economically, this reduction means that when aggregate effort rises, each player's marginal influence on the contest probability Ξ declines inversely with S^* , reflecting diminishing aggregate returns to total effort intensity.²³

Lemma 2 (Cost-Prize First-Order Cancellation). *Fix $\mathcal{L}$ and a reference equilibrium $(\tilde{a}_0, e_0^*)$ with $H(\tilde{a}_0) = 0$. Let $\Delta e := e^* - e_0^*$. Then*

$$\sum_{i \in L} a_{i,0} [h'(e_i^*) - h'_i] = -P[\Xi(e^*) - \Xi(e_0^*)] + O(\|\Delta e\|^2). \quad (.8)$$

Substituting Lemma 2 into (.5),

$$0 \approx P[\Xi(e^*) - \Xi(e_0^*)] - \sum_i (a_i - a_{i,0}) h'_i - \sum_i a_{i,0} [h'(e_i^*) - h'_i],$$

and using (.8) gives the *symmetrized* first-order balance

$$0 \approx 2P[\Xi(e^*) - \Xi(e_0^*)] - \sum_{i \in L} (a_i - a_{i,0}) h'_i + O(\|\Delta e\|^2). \quad (.9)$$

Using (.7),

$$\Xi(e^*) - \Xi(e_0^*) \approx C_0 \left(\frac{1}{S^*(\tilde{a})} - \frac{1}{S_0} \right),$$

so (.9) becomes

$$0 \approx 2PC_0 \left(\frac{1}{S^*(\tilde{a})} - \frac{1}{S_0} \right) - \sum_{i \in L} (a_i - a_{i,0}) h'_i + O(\Delta^2), \quad \Delta := \left| \frac{S^*(\tilde{a}) - S_0}{S_0} \right|. \quad (.10)$$

²³In the linear-lottery case ($f(e) = e$), the gradient $\nabla \Xi(e_0^*)$ is parallel to e_0^* , so any change in efforts decomposes neatly into a common scaling (changing total effort) and a reshuffle (redistributing effort). Only the scaling component affects Ξ to first order.

Uniform efficiency perturbation and the cost-weighted index. Under a uniform gain with $da_i/d\varepsilon = -1$ for all i , define the cost-weighted mean

$$\bar{a}_h := \frac{1}{\sum_{i \in L} h'_i} \sum_{i \in L} a_i h'_i, \quad (.11)$$

so $\sum_{i \in L} (a_i - a_{i,0}) h'_i = (\bar{a}_h - \bar{a}_{h,0}) \sum_{i \in L} h'_i$, with $h'_i := h'(e_{i,0}^*)$ frozen at baseline. Absorbing $\sum_{i \in L} h'_i$ into the normalization, (.10) yields the local cost-prize balance

$$H_1(\bar{a}_h, \varepsilon) \equiv \frac{2 P C_0}{S^*(\varepsilon)} - \bar{a}_h = O(\Delta^2). \quad (.12)$$

Let denote $\tilde{C}_0 := 2C_0$. Then

$$H_1(\bar{a}_h, \varepsilon) \equiv \frac{P \tilde{C}_0}{S^*(\varepsilon)} - \bar{a}_h = O(\Delta^2), \quad (.13)$$

where, in the partial derivative $\partial H_1 / \partial \bar{a}_h$, I hold S^* fixed as $S^*(\varepsilon)$. To first order,

$$\bar{a}_h \approx \frac{P \tilde{C}_0}{S^*(\tilde{a})}. \quad (.14)$$

Differentiating $H_1(\bar{a}_h(\varepsilon), \varepsilon) = 0$ with respect to ε (active set fixed) gives

$$-\frac{P \tilde{C}_0}{(S^*(\varepsilon))^2} \frac{dS^*}{d\varepsilon} - \frac{d\bar{a}_h}{d\varepsilon} = O(\Delta^2),$$

hence

$$\frac{d\bar{a}_h}{d\varepsilon} \approx -\frac{P \tilde{C}_0}{(S^*(\varepsilon))^2} \frac{dS^*}{d\varepsilon}. \quad (.15)$$

Cutoff response via the boundary KKT. The participation cutoff $\tilde{a}$ is also pinned down by the marginal-indifference (boundary KKT) condition (2.14). Linearizing the boundary condition and using the same inverse-in- S^* behavior for the prize term gives

$$\frac{d\tilde{a}}{d\varepsilon} \approx -\frac{\tilde{a}}{S^*(\varepsilon)} \frac{dS^*}{d\varepsilon}. \quad (.16)$$

Finally, substituting (.16) into (.15) gives

$$\frac{d\bar{a}_h}{d\varepsilon} \approx -\frac{P \tilde{C}_0}{(S^*(\varepsilon))^2} \left(-\frac{S^*(\varepsilon)}{\tilde{a}} \frac{d\tilde{a}}{d\varepsilon} \right) = \frac{P \tilde{C}_0}{S^*(\varepsilon) \tilde{a}} \frac{d\tilde{a}}{d\varepsilon}. \quad (.17)$$

Using $\tilde{C}_0 = 2C_0$ and the boundary KKT $\tilde{a} = \frac{P C_0}{S^*(\varepsilon)}$, Equation (.17) simplifies to

$$\frac{d\bar{a}_h}{d\varepsilon} \approx 2 \frac{d\tilde{a}}{d\varepsilon} \implies \left| \frac{d\tilde{a}}{d\varepsilon} \right| = \frac{1}{2} \left| \frac{d\bar{a}_h}{d\varepsilon} \right|, \quad (.18)$$

Finally, differentiating the definition of the equilibrium cutoff

$$\bar{a}_h = \frac{\sum_{i \in L(\bar{a})} a_i h'(e_i^*)}{\sum_{i \in L(\bar{a})} h'(e_i^*)},$$

with respect to the reform parameter ε , while holding the active set $L(\bar{a})$ fixed, gives

$$\frac{d\bar{a}_h}{d\varepsilon} = \frac{\left(\sum_{i \in L(\bar{a})} a_i h''(e_i^*) \frac{de_i^*}{d\varepsilon} \right) \left(\sum_{i \in L(\bar{a})} h'(e_i^*) \right) - \left(\sum_{i \in L(\bar{a})} a_i h'(e_i^*) \right) \left(\sum_{i \in L(\bar{a})} h''(e_i^*) \frac{de_i^*}{d\varepsilon} \right)}{\left(\sum_{i \in L(\bar{a})} h'(e_i^*) \right)^2}.$$

Using the definition of $\bar{a}_h = \frac{\sum_i a_i h'(e_i^*)}{\sum_i h'(e_i^*)}$, I can rewrite the numerator as

$$\sum_{i \in L(\bar{a})} h''(e_i^*) \frac{de_i^*}{d\varepsilon} \left[a_i \sum_{j \in L(\bar{a})} h'(e_j^*) - \sum_{j \in L(\bar{a})} a_j h'(e_j^*) \right] = \left(\sum_{j \in L(\bar{a})} h'(e_j^*) \right) \sum_{i \in L(\bar{a})} h''(e_i^*) (a_i - \bar{a}_h) \frac{de_i^*}{d\varepsilon}.$$

Substituting this back into the derivative expression and simplifying yields

$$\frac{d\bar{a}_h}{d\varepsilon} = \frac{\sum_{i \in L(\bar{a})} h''(e_i^*) (a_i - \bar{a}_h) \frac{de_i^*}{d\varepsilon}}{\sum_{i \in L(\bar{a})} h'(e_i^*)}.$$

Since the denominator $\sum_{i \in L(\bar{a})} h'(e_i^*) > 0$ acts only as a positive scaling term, I obtain, up to proportionality,

$$\frac{d\bar{a}_h}{d\varepsilon} \propto \sum_{i \in L(\bar{a})} (a_i - \bar{a}_h) h''(e_i^*) \frac{de_i^*}{d\varepsilon}. \quad (.19)$$